國立清華大學 104 學年度碩士班考試入學試題

系所班組別：數學系 應用數學組
考試科目（代碼）：線性代數（0202）

　　＊請在【答案卷、卡】作答

1. [10%] Let $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ be given by

$$T(z_1, z_2) = (z_1 + \overline{z_2}, z_1 - \overline{z_2})$$

where $\overline{z}$ denotes the complex conjugate of z. Is T a surjective mapping? Is T a (complex) linear transformation? Why or why not?

2. [10%] Let $T\colon \mathbb{R}^8 \to \mathbb{R}^8$ be a linear transformation. Suppose that $\dim(N(T)) = 4$ and $\dim(R(T) \cap N(T)) = 2$ where $N(T)$ denotes the kernel of T and $R(T)$ denotes the range of T. Find the ranks of T and $T \circ T$.

3. [10%] Let $\delta\colon M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ be a function such that $\delta(AB) = \delta(A)\delta(B)$ for any $A, B \in M_{2\times 2}(\mathbb{R})$. Suppose also that $\delta(\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]) \neq \delta(\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right])$.
   (1) Prove that $\delta(\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]) = 0$.
   (2) Prove that $\delta(B) = -\delta(A)$ if B is obtained by interchanging the rows of A.

4. [10%] Let V be a finite dimensional inner product space over $\mathbb{R}$, and let W be a subspace of V. Prove that $V = W \oplus W^{\perp}$ where $W^{\perp}$ denotes the orthogonal complement of W.

5. [10%] Define

$$S = \{\, A \in M_{2\times 3}(\mathbb{F}_p) \mid \operatorname{rank}(A) = 1 \,\}$$

where p is a prime number, $\mathbb{F}_p$ is the finite field of p elements, and $M_{2\times 3}(\mathbb{F}_p)$ is the set of all 2×3 matrices over $\mathbb{F}_p$. Compute the number of elements in S.

6. [10%] Compute the determinant of the 7×7 matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

7. [10%] Let T be a linear operator on the complex vector space $M_{3\times 3}(\mathbb{C})$ defined by $T(X) = AX$ for any $X \in M_{3\times 3}(\mathbb{C})$ where

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 2 & 3 \\ 1 & 4 & 4 \end{bmatrix}.$$

(1) Find the kernel of T.

(2) Find the algebraic multiplicity of each eigenvalue of T.

8. [10%] Let

$$A = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}..$$

Find matrices T, N such that $A = T + N$, T is diagonalizable, N is nilpotent, and $TN = NT$.

9. [10%] Let T be a linear operator on a finite-dimensional inner product space V (with inner product denoted by $\langle , \rangle$). The *adjoint* of T is the linear operator T^* on V such that $\langle T\mathrm{v}, \mathrm{w}\rangle = \langle \mathrm{v}, T^*\mathrm{w}\rangle$ for all $\mathrm{v}, \mathrm{w} \in V$. If T is invertible, show that T^* is also invertible and $(T^*)^{-1} = (T^{-1})^*$.

10. [10%] Find the singular value decomposition of the 2×3 real matrix

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$