國立清華大學 104 學年度碩士班考試入學試題

系所班組別：數學系　應用數學組

考試科目（代碼）：高等微積分　(0201)

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Advanced Calculus (Applied Math)

1. (13 pts) Suppose that $f : \mathbf{R} \to \mathbf{R}$ is uniformly continuous on $\mathbf{R}$, and for $n = 1, 2, 3, \cdots$, let

$$f_n(x) = f\left(x + \frac{1}{n}\right)$$

for $x \in \mathbf{R}$. Prove that $\{f_n\}$ converges uniformly on $\mathbf{R}$ to f.

2. (13 pts) Let $f : [0, 1] \to \mathbf{R}$ be continuous with $f(0) = 0$. Suppose that f is differentiable in $(0, 1)$, and that f' is an increasing function on $(0, 1)$. Prove that the function $g(x) = f(x)/x$ is also increasing on $(0, 1)$.

3. (13 pts) For $n = 1, 2, 3, \cdots$, let

$$f_n(x) = \lim_{k \to \infty} (\cos n! \pi x)^{2k} \qquad (x \in \mathbf{R}).$$

Find $\lim_{n \to \infty} f_n(x)$.

4. (13 pts) Let X and Y be metric spaces, where X is compact. If f is a continuous one-to-one mapping of X onto Y, prove that f^{-1} is a continuous mapping of Y onto X.

5. (15 pts) Define

$$f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}.$$

   (a) Let $\overrightarrow{u}$ be any unit vector in $\mathbf{R}^2$. Show that the directional derivative $(D_{\overrightarrow{u}} f)(0, 0)$ exists, and that its absolute value is at most 1.

   (b) Prove that f is not differentiable at $(0, 0)$.

6. (15 pts) Consider the vector field $\overrightarrow{F}$ on $\mathbf{R}^2$ defined by

$$\overrightarrow{F}(x, y) = (e^x \sin y, e^x \cos y)$$

and let Γ be the path $y = x^2$ joining $(0, 0)$ to $(1, 1)$ in $\mathbf{R}^2$. Evaluate the line integral $\int_\Gamma \overrightarrow{F} \cdot d\overrightarrow{s}$. Does this integral depend on the path joining $(0, 0)$ to $(1, 1)$? Explain.

7. (18 pts) Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

$$f(x, y) = (x + y, 2x + ay).$$

   (a) Calculate $Df(x, y)$ and show that $Df(x, y)$ is invertible if and only if $a \neq 2$.

   (b) Examine the image of the unit square $[0, 1] \times [0, 1]$ when $a = 1, 2$.

   (c) Find the area of the image of the unit disc $x^2 + y^2 \leq 1$ when $a = 3$.