國立清華大學 104 學年度碩士班考試入學試題

系所班組別：數學系 數學組
考試科目（代碼）：代數與線性代數（0102）

　　＊請在【答案卷、卡】作答

1. [10%] Let $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ be given by

$$T(z_1, z_2) = (z_1 + \overline{z_2}, z_1 - \overline{z_2})$$

where $\overline{z}$ denotes the complex conjugate of z. Is T a surjective mapping? Is T a (complex) linear transformation? Why or why not?

2. [10%] Let $T\colon \mathbb{R}^8 \to \mathbb{R}^8$ be a linear transformation. Suppose that $\dim(N(T)) = 4$ and $\dim(R(T) \cap N(T)) = 2$ where $N(T)$ denotes the kernel of T and $R(T)$ denotes the range of T. Find the ranks of T and $T \circ T$.

3. [10%] Let $\delta\colon M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ be a function such that $\delta(AB) = \delta(A)\delta(B)$ for any $A, B \in M_{2\times 2}(\mathbb{R})$. Suppose also that $\delta(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]) \neq \delta(\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])$.
   (1) Prove that $\delta(\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]) = 0$.
   (2) Prove that $\delta(B) = -\delta(A)$ if B is obtained by interchanging the rows of A.

4. [10%] Let V be a finite dimensional inner product space over $\mathbb{R}$, and let W be a subspace of V. Prove that $V = W \oplus W^{\perp}$ where $W^{\perp}$ denotes the orthogonal complement of W.

5. [10%] Define

$$S = \{\, A \in M_{2\times 3}(\mathbb{F}_p) \mid \operatorname{rank}(A) = 1 \,\}$$

where p is a prime number, $\mathbb{F}_p$ is the finite field of p elements, and $M_{2\times 3}(\mathbb{F}_p)$ is the set of all 2×3 matrices over $\mathbb{F}_p$. Compute the number of elements in S.

6. [10%] Let A_n denote the alternating group on a set of n elements. Construct a surjective group homomorphism $\phi\colon A_4 \to A_3$ with kernel of ϕ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

7. [10%] Prove that every finitely generated subgroup of the additive group $\mathbb{Q}/\mathbb{Z}$ is finite cyclic.

8. [10%] An element a of a ring R is called *nilpotent* if $a^n = 0$ for some positive integer n. Show that the collection of all nilpotent elements in a commutative ring R is an ideal.

9. [10%] Let $\mathbb{Z}[\sqrt{5}]$ be the integral domain of all numbers $\alpha = a + b\sqrt{5}$ with $a, b \in \mathbb{Z}$, and set $N(\alpha) = a^2 - 5b^2$.
   (1) Prove that $N(\alpha\beta) = N(\alpha)N(\beta)$ for any $\alpha, \beta \in \mathbb{Z}[\sqrt{5}]$.
   (2) Prove that α is a unit in $\mathbb{Z}[\sqrt{5}]$ if and only if $N(\alpha) = \pm 1$.

10. [10%] Let $K \subset \mathbb{C}$ be the splitting field of the polynomial $x^4 - 7$ over $\mathbb{Q}$. Find the degree of K over $\mathbb{Q}$.