

國立清華大學 104 學年度碩士班考試入學試題

系所班組別：數學系 數學組

考試科目（代碼）：高等微積分 (0101)

共 1 頁，第 1 頁 *請在【答案卷】作答

Advanced Calculus (Pure Math)

- (13 pts) If $(a, b) \subset \mathbf{R}$ is a bounded interval and $f : (a, b) \rightarrow \mathbf{R}$ is uniformly continuous, prove that f is bounded on (a, b) .
- (13 pts) Let $f : (0, 1] \rightarrow \mathbf{R}$ be differentiable, and suppose that there is a positive constant m such that $|f'(x)| \leq m$ for all $x \in (0, 1]$. Define $a_n = f(\frac{1}{n})$ for $n = 1, 2, 3, \dots$. Prove that $\lim_{n \rightarrow \infty} a_n$ exists.
- (13 pts) Investigate the continuity and differentiability of the function

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

at the point $(0, 0)$.

- (13 pts) For $n = 1, 2, 3, \dots$, let

$$f_n(x) = \lim_{k \rightarrow \infty} (\cos n! \pi x)^{2k} \quad (x \in \mathbf{R}).$$

Find $\lim_{n \rightarrow \infty} f_n(x)$.

- (13 pts) Let X and Y be metric spaces, where X is compact. If f is a continuous one-to-one mapping of X onto Y , prove that f^{-1} is a continuous mapping of Y onto X .
- (15 pts) Consider the vector field \vec{F} on \mathbf{R}^2 defined by

$$\vec{F}(x, y) = (e^x \sin y, e^x \cos y)$$

and let Γ be the path $y = x^2$ joining $(0, 0)$ to $(1, 1)$ in \mathbf{R}^2 . Evaluate the line integral $\int_{\Gamma} \vec{F} \cdot d\vec{s}$. Does this integral depend on the path joining $(0, 0)$ to $(1, 1)$? Explain.

- (20 pts) Let $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be defined by

$$f(x, y) = (x + y, 2x + ay).$$

- Calculate $Df(x, y)$ and show that $Df(x, y)$ is invertible if and only if $a \neq 2$.
- Examine the image of the unit square $[0, 1] \times [0, 1]$ when $a = 1, 2$.
- Find the area of the image of the unit disc $x^2 + y^2 \leq 1$ when $a = 3$.