# 國立清華大學 103 學年度碩士班考試入學試題

系所班組別：數學系　應用數學組

考試科目（代碼）：線性代數 (0202)

　　*請在【答案卷、卡】作答

## LINEAR ALGEBRA

1. (10%) Prove or disprove that $\mathbb{R}^3 \setminus \{(1,2,3)\}$ is a vector space.

2. (10%) Is a complex vector space always a real vector space? Is a real vector space always a complex vector space? Prove your claim or give a counterexample.

3. (10%) Suppose that $A$ is an $n \times n$ matrix satisfying $A^{100} = 0$. Show that the matrix $I_n - A$ is invertible where $I_n$ is the $n \times n$ identity matrix.

4. (10%) Let $A$ be an $m \times n$ matrix with column vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$. Suppose that $\mathbf{a}_1 + \cdots + \mathbf{a}_n = 0$, show that $rank(A) < n$.

5. (15%) Let $V = \mathscr{C}^1(0,1)$ be the vector space of continuously differentiable functions on the interval $(0,1)$. Define $T : V \to V$ by

$$T(f)(t) = tf'(t)$$

Prove that every real number is an eigenvalue of $T$ and find the corresponding eigenvectors.

6. (15%) Let $\mathrm{GL}(n,\mathbb{R})$ be the space of all $n \times n$ invertible real matrices and $\mathrm{Mat}(n,\mathbb{R})$ be the space of all $n \times n$ real matrices. Let $d$ be the metric on $\mathrm{Mat}(n,\mathbb{R})$ defined by

$$d(A,B) := \sup_{i,j=1,\ldots,n} \{|a_{ij} - b_{ij}|\}$$

where $A = [a_{ij}], B = [b_{ij}]$. Is $\mathrm{GL}(n,\mathbb{R})$ dense in $\mathrm{Mat}(n,\mathbb{R})$ under the topology induced by $d$?

7. (15%) Given two $n \times n$ matrices $A$ and $B$. Show that the characteristic polynomials of $AB$ and $BA$ are equal.

8. (15%) Suppose that $A$ is an $n \times n$ matrix with all real entries and suppose that $\lambda$ is a complex eigenvalue of $A$, with corresponding complex eigenvector $v \in \mathbb{C}^n$. Set $B = (A - \overline{\lambda} I)(A - \lambda I)$. Prove that the null space in $\mathbb{R}^n$ of $B$ is not $\{0\}$.