

國立清華大學 103 學年度碩士班考試入學試題

系所班組別：數學系 數學組

考試科目（代碼）：代數與線性代數 (0102)

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ALGEBRA AND LINEAR ALGEBRA

- (10%) Prove or disprove that $\mathbb{R}^3 \setminus \{(1, 2, 3)\}$ is a vector space.
- (10%) Is \mathbb{R}^2 with the addition $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and the multiplication $(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, y_1 y_2)$ a field?
- (10%) Suppose that A is an $n \times n$ matrix satisfying $A^{100} = 0$. Show that the matrix $I_n - A$ is invertible where I_n is the $n \times n$ identity matrix.
- (10%) Show that a finite ring with more than one element and no zero divisors is a division ring.
- (15%) Let $V = \mathcal{C}^1(0, 1)$ be the vector space of continuously differentiable functions on the interval $(0, 1)$. Define $T : V \rightarrow V$ by

$$T(f)(t) = tf'(t)$$

Prove that every real number is an eigenvalue of T and find the corresponding eigenvectors.

- (15%) Let $GL(n, \mathbb{R})$ be the space of all $n \times n$ invertible real matrices and $Mat(n, \mathbb{R})$ be the space of all $n \times n$ real matrices. Let d be the metric on $Mat(n, \mathbb{R})$ defined by

$$d(A, B) := \sup_{i,j=1,\dots,n} \{|a_{ij} - b_{ij}|\}$$

where $A = [a_{ij}]$, $B = [b_{ij}]$. Is $GL(n, \mathbb{R})$ dense in $Mat(n, \mathbb{R})$ under the topology induced by d ?

- (15%) Given two $n \times n$ matrices A and B . Show that the characteristic polynomials of AB and BA are equal.
- (15%) Let \mathbb{Q} be the field of rational numbers and R any ring. If $f, g : \mathbb{Q} \rightarrow R$ are ring homomorphisms such that $f|_{\mathbb{Z}} = g|_{\mathbb{Z}}$, show that $f = g$.