國立清華大學 102 學年度碩士班考試入學試題

系所班組別：數學系　數學組

考試科目（代碼）：高等微積分 (0101)

Advanced Calculus

1. (12 pts) A function $f : \mathbf{R} \to \mathbf{R}$ is periodic if there exists a number $p > 0$ such that $f(x+p) = f(x)$ for all $x \in \mathbf{R}$. Prove that a continuous periodic function is bounded and uniformly continuous on $\mathbf{R}$.

2. (12 pts) If $f : [a,b] \to \mathbf{R}$ is monotone increasing, prove that f is Riemann integrable on $[a,b]$.

3. (10 pts) Determine whether the curve described by the equation

$$F(x,y) = x^3 + y + \sin(xy) = 0$$

can be written in the form $y = f(x)$ in a neighborhood of $(0,0)$.

4. (20 pts)

   (a) Show that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbf{R}$.

   (b) Show that the series $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ converges on $\mathbf{R}$ to a continuous function $f(x)$.

5. (10 pts) Let $f : [a,b] \to \mathbf{R}$ be a positive and continuous function. Prove that

$$\left[\int_a^b f(x)\,dx\right]\left[\int_a^b \frac{1}{f(x)}dx\right] \geq (b-a)^2.$$

6. (12 pts) Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be differentiable at the point x_0. Prove that f has the Lipschitz property at x_0; i.e., there are constants M and $\delta_0 > 0$ such that $\|x - x_0\| < \delta_0$ implies

$$\|f(x) - f(x_0)\| \leq M\|x - x_0\|.$$

7. (12 pts) Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

$$f(x,y) = (e^x \cos y, e^x \sin y).$$

Show that f is locally invertible near every point, but is not invertible.

8. (12 pts) Let $f : [1,\infty) \to \mathbf{R}$ be continuous, and suppose that $\lim_{x\to\infty} f(x)$ exists. Use the Weierstrass approximation theorem to prove that f can be uniformly approximated on $[1,\infty)$ by a function g of the form $g(x) = p(1/x)$, where p is a polynomial.