注意：考試開始鈴響前，不得翻閱試題，
並不得書寫、畫記、作答。

國立清華大學 112 學年度碩士班考試入學試題

系所班組別：計算與建模科學研究所

科目代碼：0301

考試科目：數學分析

一作答注意事項一

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「 由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

1. Evaluate the following integrals.
   (i) $\int_0^1 \int_x^1 \sin(y^2)\, \mathrm{d}y\mathrm{d}x$ (10 points),

   (ii) $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\, \mathrm{d}y\mathrm{d}x - \int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\, \mathrm{d}x\mathrm{d}y$ (10 points).

2. Evaluate the following limits.
   (i) $\lim_{n\to\infty} \left(\frac{2^n+3^n}{2}\right)^{1/n}$ (5 points),

   (ii) $\lim_{n\to\infty} \left(\frac{2^{1/n}+3^{1/n}}{2}\right)^{n}$ (10 points).

3. Verify the maximum and minimum values of the function $f(x,y,z) = xyz$ subject to the constraint $2z^2 - 3 \leq 2y \leq 3 - x^2$ (10 points).

4. Let k be a constant, and let $f_k : [0,1] \to \mathbb{R}$ be a continuous function satisfying

$$f_k(x) + \int_0^x f_k(y)\, \mathrm{d}y - kx \int_0^1 \mathrm{e}^y f_k(y)\, \mathrm{d}y = 1, \quad \forall x \in [0,1].$$

   Use the fundamental theorem of calculus to show that:
   (i) if $k = \frac{1}{\mathrm{e}-1}$, then $f_k(x) \equiv 1$ (5 points);
   (ii) $\lim_{k\to\infty} f_k(x) = \frac{(1-\mathrm{e})\mathrm{e}^{-x}+1}{2-\mathrm{e}}$ (5 points).
   (For (i), if you assume $f_k(x) \equiv c$ a constant-valued function and obtain $c = 1$, you won't get points.)

5. Consider the function $f(x,y) = (x^2+y^2)^r$, where r is a positive constant.
   (i) Verify all r such that $\frac{\partial f}{\partial x}(x,y)$ is continuous at $(x,y) = (0,0)$ (10 points).
   (ii) Verify all r such that $\frac{\partial^2 f}{\partial x \partial y}(x,y)$ is continuous at $(x,y) = (0,0)$ (10 points).

6. **(Frobenius inner product)** Let $V = M_{n\times m}(\mathbb{C})$ be a vector space and define

$$\langle A,B\rangle_F = \text{trace}(B^*A)$$

where $B^* = \overline{B^T}$ is the conjugate transpose of B and $\text{trace}(B^*A) = \sum_{i=1}^{m}(B^*A)_{ii}$ for each $A, B \in V$.

(i) Prove that $\langle A,B\rangle_F$ is an inner product on V. (16 points.)

(ii) Define the norm of A by $\|A\|_F = \sqrt{\langle A,A\rangle_F}$. Show that

$$\|A+B\|_F^2 + \|A-B\|_F^2 = 2\|A\|_F^2 + 2\|B\|_F^2$$

for each $A, B \in V$. (8 points.)

7. Let $A \in M_{n\times n}(\mathbb{R})$ and we say that A is orthogonally equivalent to B if there exists an orthogonal matrix P such that $A = P^TBP$ and $P^TP = I_n = PP^T$ where I_n is the $n \times n$ identity matrix.

(i) Prove that A is orthogonally equivalent to a diagonal matrix if and only if A is a symmetric matrix. (16 points.)

(ii) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $A = P^TDP$. (10 points.)

(iii) Consider the quadratic equation $Q(x,y,z) = x^2 + y^2 + 2z^2 - 2yz - 2xz - 25 = 0$ in the standard coordinate system with x, y and z axes. Under a new coordinate system with x', y' and z' axes, we have

$$Q(x,y,z) = y'^2 + 3z'^2 - 25 = 0.$$

Express y' and z' as functions of the independent variables x, y and z.

(10 points.)

8. Define $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by

$$T(x,y,z) = (x + y - z, z, -x - 2y + 2z).$$

(i) Find the matrix representation M of T relative the basis

$$\beta = \{(0,-1,-1), (1,1,2), (-1,-1,-1)\}. \text{ (8 points.)}$$

(ii) Evaluate $\det(2M^{20}(M - 2I_3)^{23})$ where I_3 is the 3×3 identity matrix.

(7 points.)