國立清華大學 107 學年度碩士班考試入學試題

系所班組別：計算與建模科學研究所

考試科目（代碼）：微積分(0301)

　　　*請在【答案卷】作答

(Note: （總分 150 分）Do not change rational or constant numbers (like 1/3 or π) to decimal numbers (0.333… or 3.1415…).)

1. (10 pts.) Consider the function $y = f(x)$ which is implicitly defined by

$$x^x \cdot y^y = 1, x > 0, y > 0.$$

Find the tangent line of the graph of $y = f(x)$ at the point $(x, y) = (1, 1)$.

2. (15 pts.) Let

$$f(x) = \int_{-3x}^{3x} \left[\sqrt{t^2 + e^t} + \frac{t^{123}}{t^2 + 1} \right] dt + 1$$

be defined on $(-\infty, \infty)$.

(a) Find the derivative of $f(x)$. (8 pts.)

(b) Evaluate the derivative of $f^{-1}(x)$ at $x = 1$. (7 pts.)

3. (20 pts.) Find the limits.

(a) $\lim_{x \to 0} \left(\frac{\sin x}{x^3} - \frac{1}{x \sin x} \right)$. (10 pts.)　(b) $\lim_{x \to \frac{\pi}{2}^-} \left(1 + \frac{2}{\tan x} \right)^{\frac{1}{\left(\frac{\pi}{2} - x\right)}}$. (10 pts.)

4. (22 pts.) Consider the ellipse $C : \frac{x^2}{4} + \frac{y^2}{9} = 1$.

$\left(\text{Hint: } \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \text{ is the curvature of the parametric curve } \mathbf{r}(t). \right)$

(a) Find all the point P on the curve C at which the curvature is a minimum. (8 pts.)

(b) Find all the **center** of curvature at P. (6 pts.)

(c) Find the unit tangent vector $\mathbf{T}$ and the principle unit normal vector $\mathbf{N}$ at the point $(\sqrt{2}, \frac{3\sqrt{2}}{2})$. (8 pts.)

5. (12 pts.) Find the **shortest** and the **longest distances** from the origin to the curve formed by the intersection of $x^2 + \frac{y^2}{4} + z^2 = 1$ and $2x - y + 2z = 2$.

6. (13 pts.) Let $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{\sqrt{n}2^n}$ and $g(x) = \sum_{n=1}^{\infty} \frac{e^n(x-5)^n}{\left(1+\frac{1}{n}\right)^{n^2}}$ be two infinite series. Find the largest interval such that $f(x)$ and $g(x)$ are both convergent. (Hint: Don't forget the end points of the interval.)

7. (a) (12 pts.) Prove that the Wallis's formula.

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (n-1)}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot n}, \text{if n is odd } (n \geq 3).$$

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot n} \cdot \left(\frac{\pi}{2}\right), \text{if n is even } (n \geq 2).$$

(b) (12 pts.) Use a double integral to find the volume of the solid region bounded by the paraboloid $z = 8 - 2x^2 - y^2$ and the xy-plane.

8. (a) (10 pts.) Prove that

$$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right) + C$$

where $a \neq 0$ and C is an arbitrary constant. (Hint: $x = a\sin\theta$.)

(b) (10 pts.) Use a triple integral to find the volume of the sphere $x^2 + y^2 + z^2 = 1$ is $\frac{4}{3}\pi$.

9. (14 pts.) Let $\mathbf{F}(x, y) = (-y, x)$ be the vector field defined on the path $\mathbf{r}(t) = (x(t), y(t))$ with $x^2(t) + y^2(t) = 1$. Find the line integral $\int_C \mathbf{F}(t) \cdot d\mathbf{r}(t)$ by using the following two methods.

(a) Find the line integral directly. (7 pts.)

(b) Apply Green's theorem to the line integral. (7 pts.)