國立清華大學 106 學年度碩士班考試入學試題

系所班組別：核子工程與科學研究所甲組 (0528)

考試科目（代碼）：工程數學 (2801)

 *請在【答案卷】作答

1. Solve the following differential equations and provide general solutions of $y(x)$.

(a) $\frac{1}{\ln x}\frac{dy}{dx} = \frac{(y^2-1)}{2}$ (5%)

(b) $(x+1)^2\frac{dy}{dx} + 3xy + 3y = e^x$ (5%)

(c) $\frac{dy}{dx} = \frac{2xy^5-y^2}{4y^2-3x^2y^4+y^2\cos y}$ (5%)

(d) $x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 9y = 4x^3$ (5%)

2. Find the series solution for the following differential equation about x=0.

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (2-3x^2)y = 0\ .$$ (10%)

3. Use the Laplace transform to solve the problem

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 4\,u(t-2),$$

where $x(0) = 1,\ x'(0) = -2$, and $u(t)$ is the unit step function. (10%)

4. Suppose the matrix

$$A = \begin{bmatrix} -3 & 4 & 2 \\ 1 & 0 & -1 \\ -6 & 6 & 5 \end{bmatrix}.$$

(a) Find the determinant of A and obtain the inverse matrix A^{-1} (5%)

(b) Estimate the eigenvalues and eigenvectors of A. (5%)

5. (a) Evaluate $\int_C 3x^2y^2dx + (2x^3y - 3y^2)dy$ when C is given by $y = 3x^4 - 7x^2 - 5x$ from $(0,0)$ to $(2,10)$. (5%)

(b) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,dS$ where $\vec{F} = (x^2 + e^y\tan^{-1}z)\hat{x} + (x+2y)^2\hat{y} - (8yz + x^7)\hat{z}$ and S is the surface of the region in the first octant bounded by $z = 1 - x^2,\ z = 2 - y$. (5%)

(c) Find the Fourier series of $f(x) = \begin{cases} x^2 & -\pi < x < 0 \\ 0 & 0 \le x < \pi \end{cases}$ in the interval $(-\pi, \pi)$. (5%)

6. Consider the regular Sturm-Liouville problem:

$$\frac{d}{dx}[(1+x^2)y'] + \frac{\lambda}{1+x^2}y = 0, \quad y(0) = 0, \ y(1) = 0$$

Find the eigenvalues and eigenfuncitons fo the boundary value problem. Hint: Let $x = \tan\theta$ and then use the Chain Rule. (10%)

7. Solve the partial differential equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < 1, \qquad t > 0$$

given that $u(r,t)$ is bounded at $r = 0$, and $\left.\frac{\partial u}{\partial r}\right|_{r=1} = 0$ for $t > 0$.

Hint: Let $v(r,t) = ru(r,t)$ and solve $v(r,t)$ first. (10%)

8. (a) Find all (complex) values of $\cos^{-1}\sqrt{5}$. (5%)

(b) Evaluate $\oint_C \frac{e^{2z}}{z^4+6z^3}dz$ where C: $|z| = 1$ is a closed contour, oriented counterclockwise. (5%)

(c) Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{2x^2+1}{x^4+3x^2-4}dx$. (5%)