

國立清華大學 104 學年度碩士班考試入學試題

系所班組別：核子工程與科學研究所 (0527) 甲組(工程組)

考試科目 (代碼)：工程數學 (2701)

共 3 頁，第 1 頁 *請在【答案卷】作答

1. Solve the following initial value problem with Laplace transform.

$$y'' + 4xy' - 4y = 0, y(0) = 0, y'(0) = 6 \quad (10\%)$$

2. Obtain series solution for following ODE.

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0 \quad (14\%)$$

3. Solve for $y' + \left(\frac{x^2 + y^2 + x}{2xy}\right) = 0$ (5%)

4. Obtain solution for following initial value problem.

$$y''' + 4y'' - 3y' - 18y = 0, y(0) = 3, y'(0) = 2, y''(0) = 11 \quad (6\%)$$

5. (a) Apply Leibniz rule to check $y(x) = e^x + \int_0^x t^2 \cosh(x-t) dt$ is the solution of $y'' - y = 2x$, $y(0) = y'(0) = 1$ (5%)

(b) Find the eigenvalues and eigenvectors of $\begin{bmatrix} i & 1+i \\ -1+i & 0 \end{bmatrix}$ (5%)

(c) Evaluate the surface integral $\oint_S \vec{F} \cdot \hat{n} dS$ where $\vec{F} = e^x \hat{x} + e^y \hat{y} + e^z \hat{z}$ and S is the surface of the cube $|x| \leq 5, |y| \leq 5, |z| \leq 5$. (5%)

(d) Evaluate the line integral $\oint_C \vec{v} \cdot d\ell$ where $\vec{v} = z^2 \hat{x} + x^2 \hat{y} + y^2 \hat{z}$ and C is the circle $x = 2, y^2 + z^2 = 16$. (5%)

6. We define Fourier transform as $F\{f(x)\} = \hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x)e^{-i\omega x} dx$. A Table of Fourier transform has been provided at the end of this question sheet.

(a) Evaluate the Fourier transform $F\{5xe^{-2|x|}\}$. (5%)

(b) Evaluate the inverse Fourier transform $F^{-1}\left\{\frac{1}{\omega^2 + 6\omega + 13}\right\}$ (5%)

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(c) An infinite beam resting on an elastic foundation and subjected to a load $f(x)$ can be described by the differential equation $EIy^{(4)} + ky = f(x)$ where E, I, k are physical constants. The problem is solved using Fourier transform and Fourier convolution. The solution can be expressed as $y(x) = \int_{-\infty}^{+\infty} K(x - \xi)f(\xi)d\xi$. Find the kernel $K(x)=?$ (5%)

7. Suppose that a function $T(\rho, \varphi)$ on the spherical surface $\rho = a$ is maintained at a constant value T_0 , that the function T is harmonic throughout the regions $\rho > a$ and $\rho < a$, and that T tends to zero as $\rho \rightarrow \infty$. Find T at an arbitrary point inside the sphere ($\rho < a$). What is T outside the sphere ($\rho > a$)? You are required to solve this problem as a PDE problem and to show details of your work, including the derivation of the relevant orthogonal set of eigenfunctions to this problem. (18%) [in spherical coordinates (ρ, φ, θ) , where φ is the “cone angle” measured from the z axis,

$$\nabla^2 T = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial T}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \varphi} \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial T}{\partial \varphi} \right) + \frac{1}{\rho^2 \sin^2 \varphi} \frac{\partial^2 T}{\partial \theta^2}]$$

8. Expand the function

$$f(z) = \frac{z^2 - 2z + 2}{z - 2}$$

in a Laurent series which converges in the given annular domain $1 < |z - 1|$. (12%)

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共 3 頁，第 3 頁 *請在【答案卷】作答

Table of Fourier Transforms

$f(x)$	$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx$
1. $\frac{1}{x^2 + a^2} \quad (a > 0)$	$\frac{\pi}{a} e^{-a \omega }$
2. $H(x)e^{-ax} \quad (\text{Re } a > 0)$	$\frac{1}{a + i\omega}$
3. $H(-x)e^{ax} \quad (\text{Re } a > 0)$	$\frac{1}{a - i\omega}$
4. $e^{-a x } \quad (a > 0)$	$\frac{2a}{\omega^2 + a^2}$
5. e^{-x^2}	$\sqrt{\pi} e^{-\omega^2/4}$
6. $\frac{1}{2a\sqrt{\pi}} e^{-x^2/(2a)^2} \quad (a > 0)$	$e^{-a^2\omega^2}$
7. $\frac{1}{\sqrt{ x }}$	$\sqrt{\frac{2\pi}{ \omega }}$
8. $e^{-a x /\sqrt{2}} \sin\left(\frac{a}{\sqrt{2}} x + \frac{\pi}{4}\right) \quad (a > 0)$	$\frac{2a^3}{\omega^4 + a^4}$
9. $H(x+a) - H(x-a)$	$\frac{2 \sin \omega a}{\omega}$
10. $\delta(x-a)$	$e^{-i\omega a}$
11. $f(ax+b) \quad (a > 0)$	$\frac{1}{a} e^{ib\omega/a} \hat{f}\left(\frac{\omega}{a}\right)$
12. $\frac{1}{a} e^{-ibx/a} f\left(\frac{x}{a}\right) \quad (a > 0, b \text{ real})$	$\hat{f}(a\omega + b)$
13. $f(ax) \cos cx \quad (a > 0, c \text{ real})$	$\frac{1}{2a} \left[\hat{f}\left(\frac{\omega-c}{a}\right) + \hat{f}\left(\frac{\omega+c}{a}\right) \right]$
14. $f(ax) \sin cx \quad (a > 0, c \text{ real})$	$\frac{1}{2ai} \left[\hat{f}\left(\frac{\omega-c}{a}\right) - \hat{f}\left(\frac{\omega+c}{a}\right) \right]$
15. $f(x+c) + f(x-c) \quad (c \text{ real})$	$2\hat{f}(\omega) \cos \omega c$
16. $f(x+c) - f(x-c) \quad (c \text{ real})$	$2i\hat{f}(\omega) \sin \omega c$
17. $x^n f(x) \quad (n = 1, 2, \dots)$	$i^n \frac{d^n}{d\omega^n} \hat{f}(\omega)$
18. $\alpha f(x) + \beta g(x)$	$\alpha \hat{f}(\omega) + \beta \hat{g}(\omega)$
19. $f^{(n)}(x)$	$(i\omega)^n \hat{f}(\omega)$
20. $f(x) = \int_{-\infty}^x g(\xi) d\xi,$ where $f(x) \rightarrow 0$ as $x \rightarrow \infty$	$\hat{f}(\omega) = \frac{1}{i\omega} \hat{g}(\omega)$
21. $(f * g)(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi) d\xi$	$\hat{f}(\omega)\hat{g}(\omega)$