

國立清華大學 103 學年度碩士班考試入學試題

系所班組別：核子工程與科學研究所(0528) 甲組(工程組)

考試科目 (代碼)：工程數學 (2801)

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1. The ODE $y' + p(t)y = q(t)y^n$ (n is a constant) is called Bernoulli's equation.

(a) Give the general solution for the special cases $n=0$ and $n=1$. (5%)

(b) If n is neither 0 nor 1, the ODE can be solved by transforming $y(t)$ to

$$u(t) = y^{1-n}(t). \text{ Find the ODE for } u(t). \quad (5\%)$$

(c) Use the suggested method in (b) to solve $2tyy' + y^2 = 2t$. (10%)

2. Applying Laplace transform on the two sides of the equation $y'' + ay' + by = r(t)$ (a, b : constants) with initial conditions $y(0)$ and $y'(0)$, we obtain the subsidiary equation of the ODE: $Y(s) = [y'(0) + y(0)F(s) + R(s)]Q(s)$, where $Y(s)$ and $R(s)$ are the Laplace transform of $y(t)$ and $r(t)$, respectively. $y(t)$ can be then solved by taking the inverse Laplace transform of $Y(s)$.

(a) Determine the functions $F(s)$ and $Q(s)$ in the subsidiary equation. (5%)

(b) Use the method described above and Laplace convolution to solve the equation

$$y'' + 3y' + 2y = r(t), \quad r(t) = \begin{cases} 2 & \text{if } 2 < t < 4 \\ 0 & \text{otherwise} \end{cases}, \quad y(0) = 0, \quad y'(0) = 0. \quad (10\%)$$

3. Let $[A] = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$ be a matrix.

(a) Determine the eigenvalues and the corresponding eigenvectors of $[A]$. (5%)

(b) Find the general solution of the linear ODE system: $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = [A] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. (5%)

(c) Solve the nonhomogeneous linear ODE system:

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = [A] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}, \quad y_1(1) = -4e^{-2}, \quad y_2(1) = 0. \quad (10\%)$$

4. Let $\vec{F} = (x - 2y, 2y + x, 5z)$ be a vector field in the Cartesian space.

(a) Apply the divergence theorem to evaluate the surface integral $\oiint_S \vec{F} \cdot \hat{n} \, dA$ where $S: x^2 + y^2 + z^2 = 25$ is the surface and \hat{n} the surface normal pointing outward. (5%)

(b) Evaluate the line integral of \vec{F} along a closed curve $C: x^2 + y^2 = 4, z = -3$ (counter-clockwise as seen by a person standing at the origin) by Stokes' theorem. (5%)

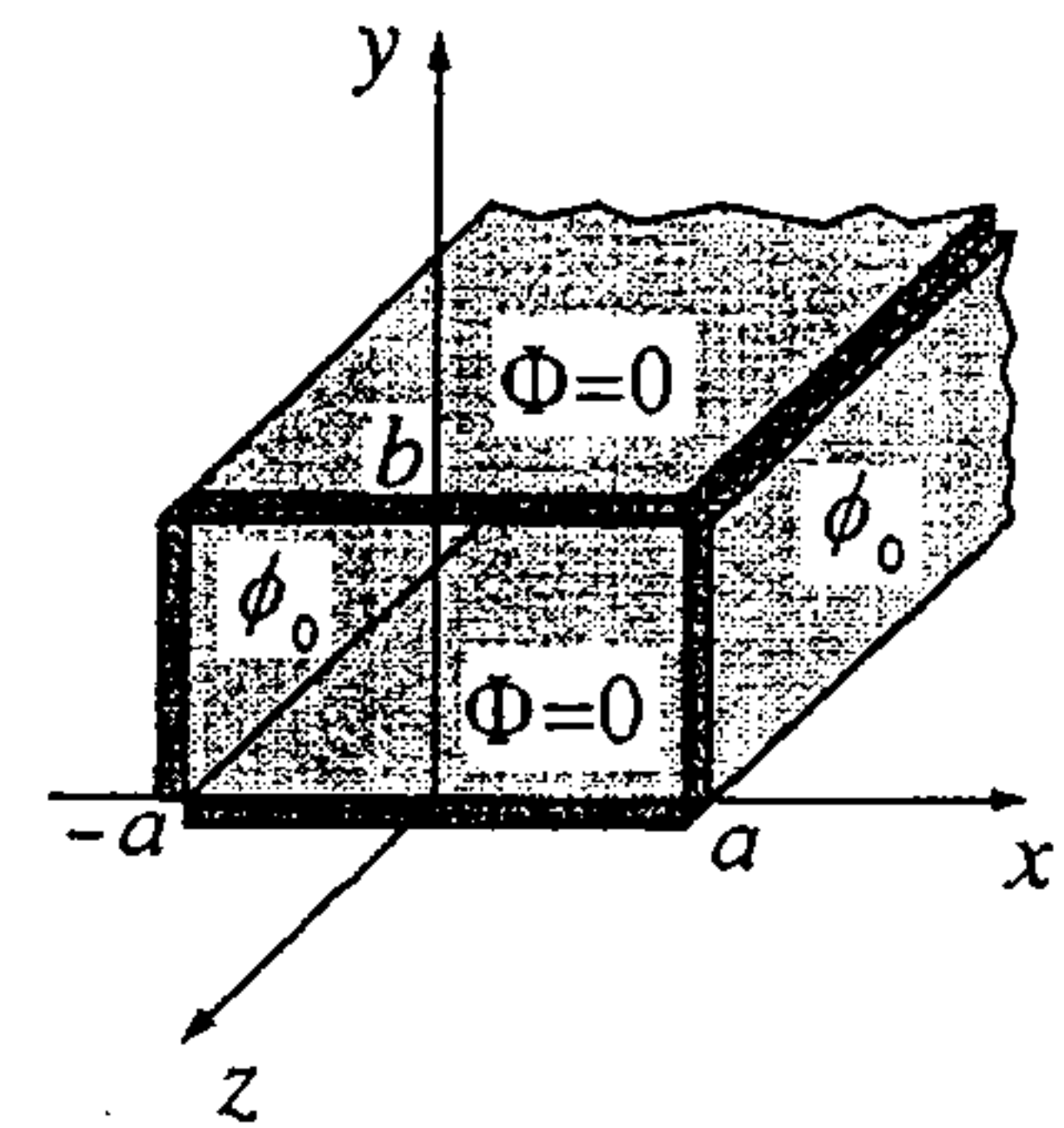
5. In this question, we solve the PDE in a 3D rectangular space,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi(x, y, z) = 0 \quad \text{with the boundary conditions:}$$

$$\Phi(\pm a, y, z) = \phi_0, \quad \Phi(x, 0, z) = 0, \quad \Phi(x, b, z) = 0 \quad \text{for } -\infty < z < \infty.$$

(a) Show, by the method of separating variables, that the eigenfunctions of the PDE take the form $u_n(x, y, z) = \cosh(k_n x) \sin(k_n y)$, $n = 1, 2, 3, \dots$. Determine $k_n = ?$ (5%)

(b) The solution can be expressed by $\Phi(x, y, z) = \sum_{n=1}^{\infty} c_n u_n(x, y, z)$. Find $c_n = ?$ (5%)



6. Let $f(z) = \frac{3z^2}{(z^2 + \frac{1}{4})(2-z)}$ be a complex function.

(a) Find the poles of $f(z)$. Determine the order of pole and the residue at each pole. (10%)

(b) Evaluate the integral $\oint_C f(z) dz$, $C: |z|=1$ (oriented counterclockwise). (5%)

7. A 2D Laplace equation $\nabla^2 \Phi(r, \theta) = 0$ on a disk of radius R for Dirichlet boundary condition can be solved by Poisson's integral formula and the solution can be expressed by a series, as

$$\Phi(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{R} \right)^n (a_n \cos n\theta + b_n \sin n\theta),$$

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$$\text{where } \begin{cases} a_0 = \frac{1}{2\pi} \int_0^{2\pi} \Phi(R, \mu) d\mu, \\ a_n = \frac{1}{\pi} \int_0^{2\pi} \Phi(R, \mu) \cos n\mu d\mu, \\ b_n = \frac{1}{\pi} \int_0^{2\pi} \Phi(R, \mu) \sin n\mu d\mu \end{cases} \quad n = 1, 2, 3, \dots$$

Given the boundary condition $\Phi(2, \theta) = \theta$ for $0 \leq \theta < 2\pi$, find the solution $\Phi(r, \theta)$ on the disk ($r \leq 2$), expanded up to the $n = 2$ term. (10%)