注意：考試開始鈴響前，不得翻閱試題，

並不得書寫、畫記、作答。

國立清華大學 111 學年度碩士班考試入學試題

系所班組別：工程與系統科學系
乙組

科目代碼：3101

考試科目：工程數學

一作答注意事項一

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

1. Solve the differential equations.

(a) $\frac{dy}{dx} + \frac{y}{x} = x^2 \ln x.$ (5%)

(b) $\frac{dy}{dx} = \frac{x(6xy-1)}{y-2x^3}$, obtain $y(x)$ that subjects to $y(1) = 1$. (5%)

(c) $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -x^{3/2}.$ (7%)

2. Find the series solution of the following differential equation about x = 0.

$$3x\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - y = 0 \ .$$ (10%)

You have to express the solution in the form of $y(x) = C_1y_1(x) + C_2y_2(x)$. To save time, you can only show the first four terms of $y_1(x)$ and $y_2(x)$.

3. Use the Laplace transform to solve the problem and obtain $x(t)$.

$$\frac{dx}{dt} + 2x + \int_0^t x(\tau)d\tau = 1 - \ u(t-1),$$ (8%)

where $x(0) = 0$ and $u(t)$ is the unit step function.

4. Consider the matrix

$$M = \begin{bmatrix} 7 & 3 & -3 \\ -2 & 1 & 2 \\ 4 & 3 & 0 \end{bmatrix}.$$

(a) Find the determinant of M and obtain the inverse matrix M^{-1}. (7%)

(b) Estimate the eigenvalues and eigenvectors of M. (8%)

5. (a) Expand the function f on the interval $(-\pi, \pi)$ in Fourier series: (5%)

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 2, & 0 \le x < \pi \end{cases}$$

(b) Determine the potential function of $\mathbf{v} = (2x + yz)\mathbf{i} + (xz + 2yz)\mathbf{j} + (xy + y^2 + 1)\mathbf{k}$. Evaluate the line integral along the curve $\mathbf{c} = (t-2)^2\mathbf{i} + (t+3)\mathbf{j} + (4t)\mathbf{k}$, $0 \le t \le 6$ (5%)

(c) Determine the surface integrals of the scalar function f and the vector function $\mathbf{F}$ on the surface $\mathbf{S}$: $\iint_S f dA = ?$ $\iint_S \mathbf{F} \cdot d\mathbf{A} = ?$ (5%)

$$\mathbf{S}(u,v) = u^2\mathbf{i} + v^2\mathbf{j} + (u^2 - 2v^2)\mathbf{k}, \quad 0 \le u \le 2, \quad 0 \le v \le 1,$$
$$f(x,y,z) = y + z$$
$$\mathbf{F}(x,y,z) = x\mathbf{i} + (x+2y)\mathbf{j} + y\mathbf{k}$$

6. Solve the displacement $u(x,t)$ of a semi-infinite elastic string by using Laplace transform: (10%)

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0, \quad (a > 0 \text{ is a constant})$$
$$u(0,t) = \sin(2t), \quad u(\infty, t) = 0, \quad t > 0$$
$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad x > 0$$

7. Solve the boundary value problem for $u(x,t)$:

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0$$

with $u(0,t) = 0$, $5u(1,t) + u_x(1,t) = 0$, $t > 0$ and
$u(x,0) = 1$, $0 < x < 1$ (10%)

8. (a) Expand $f(z) = \frac{1}{z(z-2i)}$ by a Laurent series that is valid for $1 < |z+1| < \sqrt{5}$ (5%)

(b) Determine the integral $\oint_C \left(z\exp(z^{-2}) + \frac{\sin z}{z^5 - 4z^4 + 3z^3}\right) dz$ where C is the circle of $|z| = 3/2$, oriented counterclockwise. (5%)

(c) Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{\cos x}{(x+1)(x^2-2x+2)} dx$ (5%)