注意：考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。

國立清華大學 108 學年度碩士班考試入學試題

系所班組別：**工程與系統科學系 乙組**

考試科目(代碼)：**工程數學(3101)**

—作答注意事項—

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 作答中如有發現試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「⬇由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

1. Solve the differential equations and provide general solutions of $y(x)$.

(a) $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = x(\ln x)$ (5%)

(b) $\dfrac{dy}{dx} = \dfrac{-6x^2y}{4y+8x^3}$, y(1)=1 (5%)

(c) $\dfrac{dy}{dx} = 2y - 6xy^2$ (5%)

2. Use the Laplace transform to solve the problem

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = f(t), \text{ where } f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 2, & t \ge 1 \end{cases},$$

$x(0) = 1$, and $x'(0) = -1$. (10%)

You may express $f(t)$ in terms of a unit step function.

3. Find the series solution of the following differential equation about x = 0.

$$2x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} - (x^2+2)y = 0 \ .$$ (10%)

You have to express the solution in the form of $y(x) = C_1y_1(x) + C_2y_2(x)$. To save time, you can only show the first three terms of $y_1(x)$ and $y_2(x)$.

4. Consider the matrix

$$M = \begin{bmatrix} -3 & 2 & 2 \\ -11 & 5 & 7 \\ -1 & 2 & 0 \end{bmatrix}.$$

(a) Find the determinant of A (3%) and obtain the inverse matrix A^{-1} (4%)

(b) Estimate the eigenvalues (4%) and eigenvectors of A (4%)

5. (a) Use Stokes' theorem to evaluate $\oint_C \mathbf{v}\cdot d\mathbf{R}$ where $\mathbf{v} = x^3z\hat{\mathbf{k}}$ and C is a loop starting from (1, 1, 0), to (-1, 1, 0), to (0, 0,1) and going back (1, 1, 0). (5%)

(b)Use divergence theorem to evaluate $\oint_S \mathbf{v}\cdot\hat{\mathbf{n}}dA$, where $\mathbf{v} = yz^2\hat{\mathbf{k}}$ and S is the surface of a pentahedron with vertices at (0, 0, 0), (2, 0, 0), (0, 0, 3), (2, 0, 3), (0, 4, 3), (2, 4, 3). (5%)

6. Use the method of separation of variables to solve the diffusion problem

$$\alpha^2 u_{xx} - u_t = 0 \qquad (\alpha \neq 0,\ 0 < x < L,\ 0 < t < \infty)$$

under the boundary condition:

$$u(0,t) = 5,\ u_x(L,t) = 3 \quad (0 < t < \infty),$$

$$u(x,0) = 3x + 10\sin(\tfrac{3\pi}{L}x)\cos(\tfrac{2\pi}{L}x) + 5 \quad (0 < x < L)$$ (10%)

7. Use Fourier transform to solve the heat conduction problem on an infinite rod:

$$9u_{xx} = u_t \quad (-\infty < x < \infty,\ 0 < t < \infty)$$

with $u(x,0) = 5\delta(x-10),\ -\infty < x < \infty$ [$\delta(x)$ is the Dirac's delta function]

(10%)

8. Find the solution for the Dirichlet problem on a circular disk:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \ \ (0 \le r < b), \quad u(b,\theta) = \begin{cases} 10, & 0 \le \theta < \pi \\ 0, & \pi \le \theta < 2\pi \end{cases}.$$

Here we assume that $u(r,\theta)$ is bounded at $r = 0$ and $u(r,\theta) = u(r,\theta + 2\pi)$.

(10%)

9. (a) Evaluate the integral $\oint_C \frac{z}{(z^2+i)^2} dz$ where C is the counterclockwise circle $|z| = 5$. (5%)

(b)Find the Taylor or Laurent expansions of $f(z) = \frac{1}{(z+i)(z-2i)}$ about $z = 0$ in the circle $|z| < 1$ and in the annulus $2 < |z| < \infty$, respectively. (5%)