國立清華大學 107 學年度碩士班考試入學試題

系所班組別：工程與系統科學系碩士班 乙組(0529)

考試科目（代碼）：工程數學 (2901)

　　*請在【答案卷】作答

1. Solve the differential equations and provide general solutions of $y(x)$.

   (a) $\frac{dy}{dx} + 3x^2y = x^5$ (5%)

   (b) $(y - x^2y)\frac{dy}{dx} = x + 1$ (5%)

   (c) $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = (\ln x)x^3$ (5%)

2. Use the Laplace transform to solve the problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 6y = f(t), \text{ where } f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases},$$

$y(0) = 1$, and $y'(0) = -1$. (10%)

You may express $f(t)$ in terms of the unit step function.

3. Find the series solution of the following differential equation about x=0.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right)y = 0 \ .$$ (10%)

You have to express the solution in the form of $y(x) = C_1y_1(x) + C_2y_2(x)$. To save time, you can only show the first three terms of $y_1(x)$ and $y_2(x)$.

4. Suppose the matrix

$$M = \begin{bmatrix} 7 & -2 & -1 \\ 3 & 0 & 1 \\ 9 & -2 & -3 \end{bmatrix}.$$

(a) Find the determinant of A (3%) and obtain the inverse matrix A^{-1} (4%).

(b) Estimate the eigenvalues (4%) and eigenvectors of A (4%).

5. (a) Evaluate $\oint_C 3zdx + 2xdy + ydz$, where C is the trace of the cylinder $x^2 + y^2 = 4$ in the plane $y + z = 6$. Orient C counterclockwise as viewed from above. (5%)

(b) Let S be the surface of the region bounded by the hemisphere $(x-3)^2 + (y-2)^2 + (z-1)^2 = 25,\ 1 \le z \le 6$, and the plane $z = 1$.

Evaluate $\oiint_S (\mathbf{F} \cdot \mathbf{n}) dS$ where $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j} - 5(z-1)\mathbf{k}$. (5%)

6. Use Fourier transform $\tilde{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{ikx}dx$ to solve the heat equation $\alpha \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $-\infty < x < \infty$, $t > 0$, subject to $u(x,0) = \begin{cases} u_0, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ (10%)

7. Solve the nonhomogeneous boundary value problem: $2\frac{\partial^2 u}{\partial x^2} + 4 = \frac{\partial u}{\partial t}$ with $u(0,t) = 0, u(1,t) = 1$ for $t > 0$, and $u(x,0) = -x^2$ for $0 < x < 1$.
Hint: Assume that $u(x,t) = v(x,t) + \psi(x)$. If one sets $\psi'' + 2 = 0$, the nonhomogeneous PDE can reduce to a homogeneous PDE. (10%)

8. Evaluate $\oint_C \frac{1}{(z+1)^2(z-2)} dz$, where C is the ellipse $\frac{x^2}{9} + y^2 = 1$ and the loop orientation is counterclockwise. (10%)

9. A flow in a corner, sketched in the figure (left), satisfies the Laplace equation $\nabla^2\phi = 0$ with the boundary condition $\nabla\phi \cdot \hat{\mathbf{n}} \equiv \frac{\partial \phi}{\partial n} = 0$ where $\phi(x,y)$ is the scalar velocity potential. Use the conformal mapping $w = z^2$ to determine $\phi(x,y)$ and the velocity field $\mathbf{q} = \nabla\phi$. Hint: After the conformal mapping, the flow becomes a uniform stream (refer to the figure sketched on the right). We

assume that the uniform stream has speed U_0. (10%)

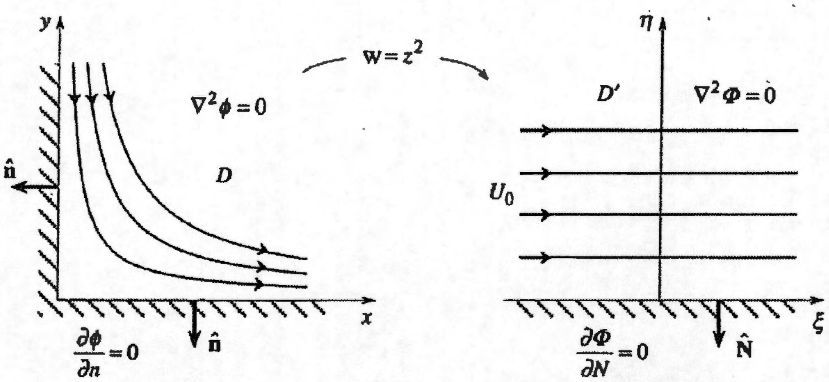

Figure Corner flow conformal map