國立清華大學 104 學年度碩士班考試入學試題

系所班組別：工程與系統科學系 (0526) 乙組

考試科目（代碼）：工程數學 (2601)

　　*請在【答案卷】作答

1. Solve the following initial value problem with Laplace transform.

$$y'' + 4xy' - 4y = 0,\ y(0) = 0,\ y'(0) = 6 \qquad (10\%)$$

2. Obtain series solution for following ODE.

$$x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0 \qquad (14\%)$$

3. Solve for $y' + \left(\frac{x^2+y^2+x}{2xy}\right) = 0$ (5%)

4. Obtain solution for following initial value problem.

$$y''' + 4y'' - 3y' - 18y = 0,\ y(0) = 3,\ y'(0) = 2,\ y''(0) = 11 \qquad (6\%)$$

5. (a) Apply Leibniz rule to check $y(x) = e^x + \int_0^x t^2 \cosh(x-t)\,dt$ is the solution of $y'' - y = 2x,\ y(0) = y'(0) = 1$ (5%)

(b) Find the eigenvalues and eigenvectors of $\begin{bmatrix} i & 1+i \\ -1+i & 0 \end{bmatrix}$ (5%)

(c) Evaluate the surface integral $\oint_S \vec{F} \cdot \hat{n}\, dS$ where $\vec{F} = e^x\hat{x} + e^y\hat{y} + e^z\hat{z}$ and S is the surface of the cube $|x| \le 5,\ |y| \le 5,\ |z| \le 5$. (5%)

(d) Evaluate the line integral $\oint_C \vec{v} \cdot d\ell$ where $\vec{v} = z^2\hat{x} + x^2\hat{y} + y^2\hat{z}$ and C is the circle $x = 2,\ y^2 + z^2 = 16$. (5%)

6. We define Fourier transform as $F\{f(x)\} = \hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x)e^{-i\omega x}\,dx$. A Table of Fourier transform has been provided at the end of this question sheet.

(a) Evaluate the Fourier transform $F\{5xe^{-2|x|}\}$. (5%)

(b) Evaluate the inverse Fourier transform $F^{-1}\left\{\frac{1}{\omega^2+6\omega+13}\right\}$ (5%)

(c) An infinite beam resting on an elastic foundation and subjected to a load $f(x)$ can be described by the differential equation $EIy^{(4)} + ky = f(x)$ where E, I, k are physical constants. The problem is solved using Fourier transform and Fourier convolution. The solution can be expressed as $y(x) = \int_{-\infty}^{+\infty} K(x-\xi)f(\xi)d\xi$. Find the kernel $K(x)$=? (5%)

7. Suppose that a function $T(\rho, \varphi)$ on the spherical surface $\rho = a$ is maintained at a constant value T_0, that the function T is ***harmonic*** throughout the regions $\rho > a$ and $\rho < a$, and that T tends to zero as $\rho \to \infty$. Find T at an arbitrary point inside the sphere ($\rho < a$). What is T outside the sphere ($\rho > a$)? You are required to solve this problem as a PDE problem and to show details of your work, including the derivation of the relevant orthogonal set of eigenfunctions to this problem. (18%)
[in spherical coordinates (ρ, φ, θ), where φ is the "cone angle" measured from the z axis,

$$\nabla^2 T = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 \frac{\partial T}{\partial \rho}) + \frac{1}{\rho^2 \sin\varphi}\frac{\partial}{\partial \varphi}(\sin\varphi \frac{\partial T}{\partial \varphi}) + \frac{1}{\rho^2 \sin^2\varphi}\frac{\partial^2 T}{\partial \theta^2}$$]

8. Expand the function

$$f(z) = \frac{z^2 - 2z + 2}{z - 2}$$

in a Laurent series which converges in the given annular domain $1 < |z-1|$. (12%)

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Table of Fourier Transforms

	$f(x)$	$\hat{f}(\omega)=\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$
1.	$\dfrac{1}{x^2+a^2}\quad (a>0)$	$\dfrac{\pi}{a}e^{-a\lvert\omega\rvert}$
2.	$H(x)e^{-ax}\quad (\mathrm{Re}\,a>0)$	$\dfrac{1}{a+i\omega}$
3.	$H(-x)e^{ax}\quad (\mathrm{Re}\,a>0)$	$\dfrac{1}{a-i\omega}$
4.	$e^{-a\lvert x\rvert}\quad (a>0)$	$\dfrac{2a}{\omega^2+a^2}$
5.	e^{-x^2}	$\sqrt{\pi}e^{-\omega^2/4}$
6.	$\dfrac{1}{2a\sqrt{\pi}}e^{-x^2/(2a)^2}\quad (a>0)$	$e^{-a^2\omega^2}$
7.	$\dfrac{1}{\sqrt{\lvert x\rvert}}$	$\sqrt{\dfrac{2\pi}{\lvert\omega\rvert}}$
8.	$e^{-a\lvert x\rvert/\sqrt{2}}\sin\left(\dfrac{a}{\sqrt{2}}\lvert x\rvert+\dfrac{\pi}{4}\right)\quad (a>0)$	$\dfrac{2a^3}{\omega^4+a^4}$
9.	$H(x+a)-H(x-a)$	$\dfrac{2\sin\omega a}{\omega}$
10.	$\delta(x-a)$	$e^{-i\omega a}$
11.	$f(ax+b)\quad (a>0)$	$\dfrac{1}{a}e^{ib\omega/a}\hat{f}\left(\dfrac{\omega}{a}\right)$
12.	$\dfrac{1}{a}e^{-ibx/a}f\left(\dfrac{x}{a}\right)\quad (a>0,\ b\ \text{real})$	$\hat{f}(a\omega+b)$
13.	$f(ax)\cos cx\quad (a>0,\ c\ \text{real})$	$\dfrac{1}{2a}\left[\hat{f}\left(\dfrac{\omega-c}{a}\right)+\hat{f}\left(\dfrac{\omega+c}{a}\right)\right]$
14.	$f(ax)\sin cx\quad (a>0,\ c\ \text{real})$	$\dfrac{1}{2ai}\left[\hat{f}\left(\dfrac{\omega-c}{a}\right)-\hat{f}\left(\dfrac{\omega+c}{a}\right)\right]$
15.	$f(x+c)+f(x-c)\quad (c\ \text{real})$	$2\hat{f}(\omega)\cos\omega c$
16.	$f(x+c)-f(x-c)\quad (c\ \text{real})$	$2i\hat{f}(\omega)\sin\omega c$
17.	$x^n f(x)\quad (n=1,2,\ldots)$	$i^n\dfrac{d^n}{d\omega^n}\hat{f}(\omega)$
18.	$\alpha f(x)+\beta g(x)$	$\alpha\hat{f}(\omega)+\beta\hat{g}(\omega)$
19.	$f^{(n)}(x)$	$(i\omega)^n\hat{f}(\omega)$
20.	$f(x)=\int_{-\infty}^{x} g(\xi)\,d\xi$, where $f(x)\to 0$ as $x\to\infty$	$\hat{f}(\omega)=\dfrac{1}{i\omega}\hat{g}(\omega)$
21.	$(f*g)(x)=\int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi$	$\hat{f}(\omega)\hat{g}(\omega)$