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Note: Detailed derivations are required to obtain a full score for each problem.

1. (15 pts) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & a \end{pmatrix}$, $x \in \mathsf{M}_{3\times 1}(\mathbb{R})$, and $b = \begin{pmatrix} 100 \\ 200 \\ c \end{pmatrix}$.

(a) (5%) Find the conditions for a such that the system of equations $Ax = b$ has a unique solution.

(b) (5%) Find the conditions for a and c such that $Ax = b$ has infinitely many solutions.

(c) (5%) If $Ax = b$ has infinitely many solutions, is it possible to find a positive integer n such that $A^n x = b$ has a unique solution? Why or why not?

2. (10 pts) Let V be the vector space spanned by the set of functions $\{1, \cos\omega t, \sin\omega t\}$, defined on the time domain $t \in \mathbb{R}$. Assume that the angular frequency $\omega \geq 0$, and let $\beta = \{1, \cos\omega t, \sin\omega t\}$ be regarded as an ordered basis for V. Define a linear transformation $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ as follows,

$$\mathsf{T}(x(t)) = m\frac{d^2x(t)}{dt^2} + r\frac{dx(t)}{dt} + kx(t),$$

where parameters m, r, k are non-negative.

(a) (5%) Find the matrix representation $A = [\mathsf{T}]_\beta$.

(b) (5%) If $r = 0, m \neq 0$, find the condition for ω such that $\dim(\mathsf{N}(\mathsf{T})) > 0$.

3. (9 pts) Let V be an inner product space and let T be a linear operator on V. Prove or disprove the following statement.

$$\mathsf{R}(\mathsf{T}^*)^\perp = \mathsf{N}(\mathsf{T}).$$

4. (16 pts) Given the Schur decomposition theorem as follows.

Theorem 1 *Let T be a linear operator on a finite-dimensional inner product space V. If* $\det(T - tI_V)$ *splits, then there exists an orthonormal basis* β *for V such that* $[T]_\beta$ *is upper triangular.*

Use this theorem to prove or disprove the following statements.

(a) (8%) Let T be a normal operator on a finite-dimensional inner product space V over $\mathbb{C}$. Then there exists an orthonormal basis β for V such that $[\mathsf{T}]_\beta$ is diagonal.

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(b) (8%) Let T be a self-adjoint operator on a finite-dimensional inner product space V over $\mathbb{R}$. Then there exists an orthonormal basis β for V such that $[\mathsf{T}]_\beta$ is diagonal.

5. (20 pts) Based on the descriptions below:

1. Two water tanks, denoted as T_1 and T_2, are mutually connected through two pipelines.
2. Initially tanks T_1 and T_2 contain 100 liters of water each.
3. In tank T_1 water is pure; while 150 grams of salt are dissolved in tank T_2.
4. By stirring to keep the mixture uniform and circulating liquid through two pipelines at a rate of 2 liters per minute, the amounts of salt $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t.

Write down the differential equations for these two mixing tanks, and **How long** the tank T_1 will contain at least half as much salt as there will be left in tank T_2?

6. (10 pts) Construct a Fourier series over the interval of $(-\pi, \pi)$ and use Parseval identity to calculate the summation of an infinite series:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}.$$

7. (20 pts) Complex Variables:

(a) (5%) Prove or disapprove that e^{iz} is an entire function.

(b) (5%) Is $Re[\oint f(z)\,dz] = \oint Re[f(z)]\,dz$? Explain.

(c) (5%) Given that $Ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots$ for $|z| < 1$, find the **first three** terms of the Taylor series of $Ln(2z)$ with the center at $-2i$. Also find its radius of convergence.

(d) (5%) Evaluate

$$\oint_C \frac{\exp(-z)}{\cos(z)}\,dz,$$

$C : |z| = 2$, counter clockwise.

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