1. Let the joint probability density function of random variables X and Y be

$$f(x,y) = \begin{cases} c(x^2 + y^2) & \text{if } 0 \le x \le 2 \text{ and } 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Determine the value of c. (5%)

   (b) Find the marginal probability function $f_X(x)$, $f_Y(y)$ and Calculate $E(X)$ and $E(Y)$. (5%)

   (c) Find conditional probability distribution function $f_{X|Y}(x|y)$, $f_{Y|X}(y|x)$ and evaluate $E(Y|X=1)$

   (5%)

2. Let X and Y be two independent random variables

   (a) Show that $X - Y$ and $X + Y$ are uncorrelated if and only if $\text{Var}(X) = \text{Var}(Y)$. (5%)

   (b) Show that $\text{Cov}(X, XY) = E(Y)\text{Var}(X)$ (5%)

3. Let A be a 4×5 matrix. After a series of elementary row operations, we obtain a matrix: B

$$= \begin{pmatrix} 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ from } A. \text{ (Total 25 \%)}$$

   (a) Find matrix A if the 1st, 3rd, and 5th columns of A are (1, 0, 1, 0), (0, -4, 4, 4), and (1, -1, 0, 2), respectively. (6 %)

   (b) If W is generated by the column vectors of A, find an orthonormal basis for A. (6 %)

   (c) For a vector $x = (1, -1, 3, 3)$, find the shortest distance from x to W and the corresponding closet vector to x on W. (5 %)

   (d) For a linear equation system $Ax = b$, find a basis for the vector space spanned by the solutions of its homogeneous system. (4 %)

   (e) Following (d), if (-4, 0, 1, 3, -1) is in the system's solution set, find its minimal solution. (4 %)

4. Please answer the following questions: (Total: 13%)

   (a) We have four boxes. Box 1 contains 1000 components of which 5 percent are defective. Box 2 contains 500 components of which 40 percent are defective. Both boxes 3 and 4 contain 1000 components, and each has 10 percent defective. We choose at random one of the four boxes and select at random a single component from the chosen box. What is the probability that the selected component is defective? (3%)

   (b) Trains X and Y arrive at a station at random between 8 A.M. and 8:20 A.M. Train X stops for four minutes.

and train Y stops for five minutes. Assume that the trains arrive independently of each other. What is the probability that the trains meet at the station? (4%)

(c) Given three independent random variables, each uniformly distributed over the interval 0 to 1. What is the probability that they are within a distance of 1/2 of each other? (6%)

5. A fine needle of length $2a$ is dropped at random on a board covered with parallel vertical lines with distance $2b$ apart and parallel horizontal lines with distance $2c$ apart, where $b>a$ and $c>a$ as shown in Fig. 1. We shall determine the probability that the needle intersects one of the lines. Denote by random variable X the distance from the center of the needle to the nearest vertical line and by random variable θ the angle between the needle and the direction of the horizontal lines. (Total: 12%)

(a) What are the probability density functions (pdf) of X and θ? (2%)

(b) Use X, θ, a, b, c to express the condition that the needle intersects a vertical line. (2%)

(c) Use the result of (b) to determine the probability that the needle intersects one of the vertical lines. (3%)

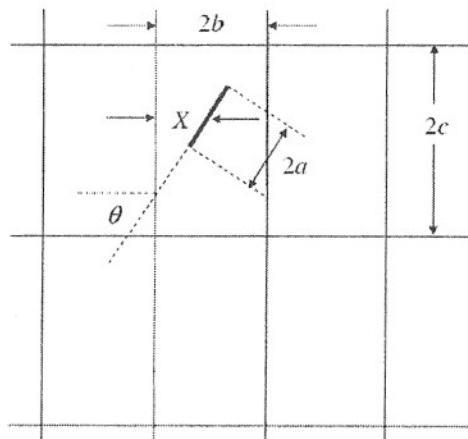(d) What is the probability that the needle intersects any of the vertical and horizontal lines? (5%)

Fig. 1

6. Let V and W be vector spaces over a field F of dimensions m and n, respectively. Let Z be a vector space given by $Z = \{(v, w): v \in V \text{ and } w \in W\}$. What is the dimension of Z (you need to justify your answer)? (10%)

7. Let $T: P_2(R) \rightarrow P_3(R)$ be given by $T(f(x)) = xf(x) + f'(x)$. (Total 15%)

(a) What is the null space $N(T)$ of T (you need to justify your answer)? (5%)

(b) What is the range $R(T)$ of T (you need to justify your answer)? (5%)

(c) Is T one-to-one? Is T onto? (you need to justify your answer) (5%)