國立清華大學命題紙

九十三學年度 電機工程學系（所） 甲、乙組 碩士班入學考試

科目 工程數學 科號 2601、2701 共 3 頁第 1 頁 *請在試卷(答案卷)內作答

Problem 1 (12%)

(a) Is the statement " For arbitrary n, any $n \times n$ real matrix must have at least one real eigenvalue" true? If yes, prove it. If not, give a counter-example. (4%)

(b) **A** is a given n x n real matrix, and we know that $\mathbf{A}\underline{x} = \underline{0}$ has nonzero solutions. ($n \geq 2$). What can we say about its eigenvalues? (2%)

(c) Continued from (b), Could **A** possibly have n linearly independent columns? (You only need to give answer in terms of "Yes" or "No") (2%)

(d) Consider $\mathbf{A}\underline{x} = \underline{b}$, where **A** is the same as that in (b) and $\underline{b}$ is an n dimension nonzero real vector. If we know that $\underline{z}$ is one solution of the system of linear equations, $\mathbf{A}\underline{x} = \underline{b}$, show that there must be some other solution $\underline{y}$ of $\mathbf{A}\underline{x} = \underline{b}$, where $\underline{y} \neq \underline{z} \neq \underline{0}$. (4%)

Problem 2 (8%)

A is an $n \times n$ real symmetric matrix and there exists another $n \times n$ real non-singular matrix **B** such that $\mathbf{A} = \mathbf{B}^T\mathbf{B}$.

(a) Show that $\underline{x}^T \mathbf{A}\, \underline{x} \geq 0$ for any n dimensional real vector $\underline{x}$, and the equality holds if and only if $\underline{x} = \underline{0}$. (4%)

(b) At what $\underline{x}$ will the function $f(\underline{x}) = (\underline{x}-\underline{a})^T \mathbf{A}^T (\underline{x}-\underline{a}) + c$ achieve its extremum, where $\underline{a}$ is a given real $n \times 1$ vector and c is a real number ? (2%)

(c) Continued from (b), whether the extremum is a maximum or a minimum, and what is the value of it ? (2%)

Problem 3 (10%)

Due to the potentially huge impact on our living society of the quantum communications and quantum computations, a research-oriented club on quantum information processing is started in a well-known university in Hsinchu. The club has two types of members---students and faculty, and the membership automatically terminates after one year. Student members of the club are so enthusiastic that they recruit other members right before their membership expires; however, faculty members never recruit because they are too busy. A student recruiter will successfully recruit two students with probability 1/2, one student and one faculty with probability 1/3, and two faculty with probability 1/6. Assume that the club was started by one student. For example, after the first year, the club either has two student members, or one student member and one faculty member, or two faculty members. Note that if the student wishes to stay in the club after one year, then he may choose to recruit himself right before his membership expires.

(a) After n years, what is the probability that no faculty will have yet been recruited? (5%)

(b) After n years, what is the probability that exactly one faculty will have been recruited and this faculty is recruited right before the end of the k-th year after the club was started, where $1 \leq k \leq n$? (5%)

Problem 4 (10%)

Let $X_0, X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) discrete random variables with the same probability mass function (pmf) $\Pr\{X_0 = i\} = p_i,\ 1 \le i \le m$. Let N be the random variable that represents the waiting time to the next occurrence of X_0, namely, N is the smallest positive integer n such that $X_0 = X_n$.

(a) Derive the pmf of the discrete random variable N. (5%)

(b) Calculate the mean, E[N], and the variance, Var(N), of N. (You will be surprised by the fact that the mean waiting time is insensitive to the pmf of X_0.) (5%)

Problem 5 (4%)

Find the probability that a poker hand contains a full house, that is three of one kind and two of the other kind ?

Problem 6 (8%)

(a) Find all solutions of $e^z = 1$, where z is a complex variable.

(b) Show that $\cos \bar{z} = \overline{\cos z}$ for all z.

(Note: $\bar{z}$ is the complex conjugate of z, and $\overline{\cos z}$ is the complex conjugate of $\cos z$.)

Problem 7 (8%)

(a) Find the all values of $\ln z$ where $z = (\sqrt{2} + i\sqrt{2})$

(b) Find the principal value of $(1+i)^{1-i}$

Problem 8 (10%)

For the following equation,

$$\frac{d^2 y}{dx^2} - e^{2x} y = 0$$

please find the solution based on the power series method and write out first 5 non-zero terms in the solution.

Problem 9 (10%)

Please find the Laplace transform of $e^{-3t} f(t)$, where

$$f(t) = \begin{cases} 0, & t < 6 \\ t^2 - 3, & t \ge 6 \end{cases}$$

Problem 10 (20%)

We have an electrical device that can be modeled as an inductor in series with a resistor. We would like to know the values of the inductor (L) and the resistor (R). However, the only possible way to obtain information is to apply an external voltage source (also in series with R and L) and observe the current through the voltage source. An engineer, Tom, was assigned for this task. Tom used an exponentially decaying voltage source ($E(t) = e^{-t}$) and he measured the current for three times. The data obtained were listed in the following table:

Time t	$E(t)$	$\frac{dI(t)}{dt}$	$I(t)$
0	1	0.5	0
1	0.0368	0.065	0.24
2	0.135	-0.05	0.23

Tom plugged in the values into the differential equation that describes the behavior of the system

$$L\frac{dI}{dt} + RI = E(t).$$

He found that in this equation there are only two unknown variables and there are three equations. Now he doesn't know how to solve this problem, since the math he learned says a system of linear equations is solvable only when the number of unknowns matches the number of equations. Tom needs your help.

(a) Please list the linear equations in matrix form of $Ax=b$, where $x=[L, R]^T$. (2%)

(b) Then Tom takes the equations to Jerry. Jerry said "why don't you just drop the last data at time 2 and use matrix inversion to solve the problem". Please follow Jerry's advice and calculate $x=A^{-1}b$. Does this give correct answers? If not, why? (5%)

(c) Later Tom asked a math guru, Elaine, about this problem. Elaine said "there must be an error in your equations and now you can simply use least square approximation to minimize the effects from the errors". Please derive a matrix equation of least square approximation in terms of A^T, A, x and b. And solve the equation to derive L and R. (5%)

(d) Finally, Tom found the source of error and he corrected the $E(t)|_{t=1}$ to 0.368. He tried to solve the equations again. After all the experience, he decided to learn more about matrices. He learned that this problem can also be solved by $x=A^+ b$, where A^+ is pseudo-inverse of A. He also know that A^+ is to take the inverse of the singular value decomposition (SVD) of A, where $A=USV^T$. $A^+ = VS^+U$, where S^+ is defined by inverting the nonzero values in S and then transpose the obtained matrix, e.g., $S_{ij}^+ = 1/S_{ji}$ (S_{ij} is the component of matrix S at i-th row and j-th column). And by a very powerful math package, octave, he obtained the matrix of U, S and V as follows.

$$U = \begin{bmatrix} -0.98 & 0.05 & 0.17 \\ -0.15 & -0.71 & -0.68 \\ 0.08 & -0.7 & 0.7 \end{bmatrix} S = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \\ 0 & 0 \end{bmatrix} V = \begin{bmatrix} -0.99 & V_{12} \\ -0.036 & V_{22} \end{bmatrix}$$

Please calculate the unknown (V_{12} and V_{22}) in V and $x=A^+b$. (5%)

(e) Having obtained the L and R from the last attempt (in the above problem d.), Tom would like to solve the original differential equation. Please round L and R to the nearest integers and solve the ODE problem. What is the $I(t)$? (3%)