九十二學年度 電機工程學 系（所） 甲、乙 組碩士班研究生招生考試

科目 工程數學 科號 2301、2401 共 3 頁第 1 頁 *請在試卷【答案卷】內作答

1. True or false. You should give reasons or counterexamples, otherwise no credits.

   (a) (3%) Let A, B, C be three $n \times n$ matrices. If A is similar to B and B is similar to C, then A is similar to C.

   (b) (3%) If A and B are diagonalizable, then $A + B$ is also diagonalizable.

   (c) (3%) There is a 3×5 matrix A with $\text{rank}(A) = 3$ and a 5×3 matrix B with $\text{rank}(B) = 3$ such that AB is the zero matrix.

   (d) (3%) Let R^{12} be the vector space of all real 12-tuples. There is a subspace V of R^{12} with $\dim(V) = 5$ so that its orthogonal complement $V^{\perp}$ has dimension 5 too.

2. Let U and W be two subspaces of a finite-dimensional vector space V. Let $U + W$ be the set of all vectors in V of the form $\boldsymbol{u} + \boldsymbol{w}$ with $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$.

   (a) (4%) Please show that $U + W$ and $U \cap W$ are subspaces of V.

   (b) (5%) Please show that

   $$\dim(U) + \dim(W) = \dim(U \cap W) + \dim(U + W).$$

3. Let M_n be the $n \times n$ matrix with 3's on the main diagonal except the first entry which is 1, 1's directly above the main diagonal and (-1)'s directly below the main diagonal, and 0's elsewhere. For example,

$$M_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 3 & 1 & 0 & 0 \\ 0 & -1 & 3 & 1 & 0 \\ 0 & 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}.$$

We define $D_n = \det M_n$.

   (a) (2%) Find D_1 and D_2.

   (b) (3%) If $n \geq 3$, find a formula for D_n in terms of D_{n-1} and D_{n-2}.

   (c) (4%) Find a formula for D_n in terms of n.

4. The covariance $Cov(X, Y)$ of two random variables X and Y is defined as

$$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

where μ_X and μ_Y are the means of X and Y respectively.

(a) (7%) Please show that if X and Y are statistically independent, then

$$Cov(X, Y) = 0.$$

(b) (8%) Please show that if X and Y have the same distribution, then

$$Cov(X - Y, X + Y) = 0.$$

5. (15%) A certain cancer is found in one person in 100. If a person does have the disease, in 99% of the cases the patient will show symptom Z. However, if a person does not have the cancer, 10% will still have the symptom Z. Determine the probability that a person with symptom Z has the cancer. Is the probability high or low?

6. (5%) Is $f(z) = (z + \bar{z})$ differentiable for all z (Yes/No)? Prove your answer.

7. (5%) Is $f(z) = |z|^2$ analytic at $z = 0$ (Yes/No)? Prove your answer.

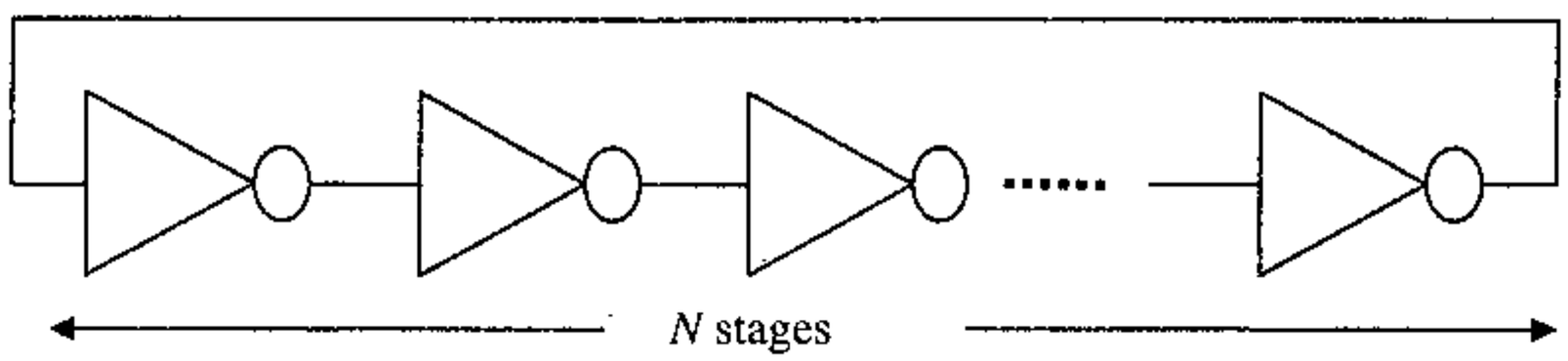
8. (5%) Show all possible z such that $\sin(z) = i \sinh(1)$.

9. (5%) Let z be a complex number that satisfies $z \neq 1, z^3 = 1$. Find the value of $(1 + z + 2z^2)^9$.

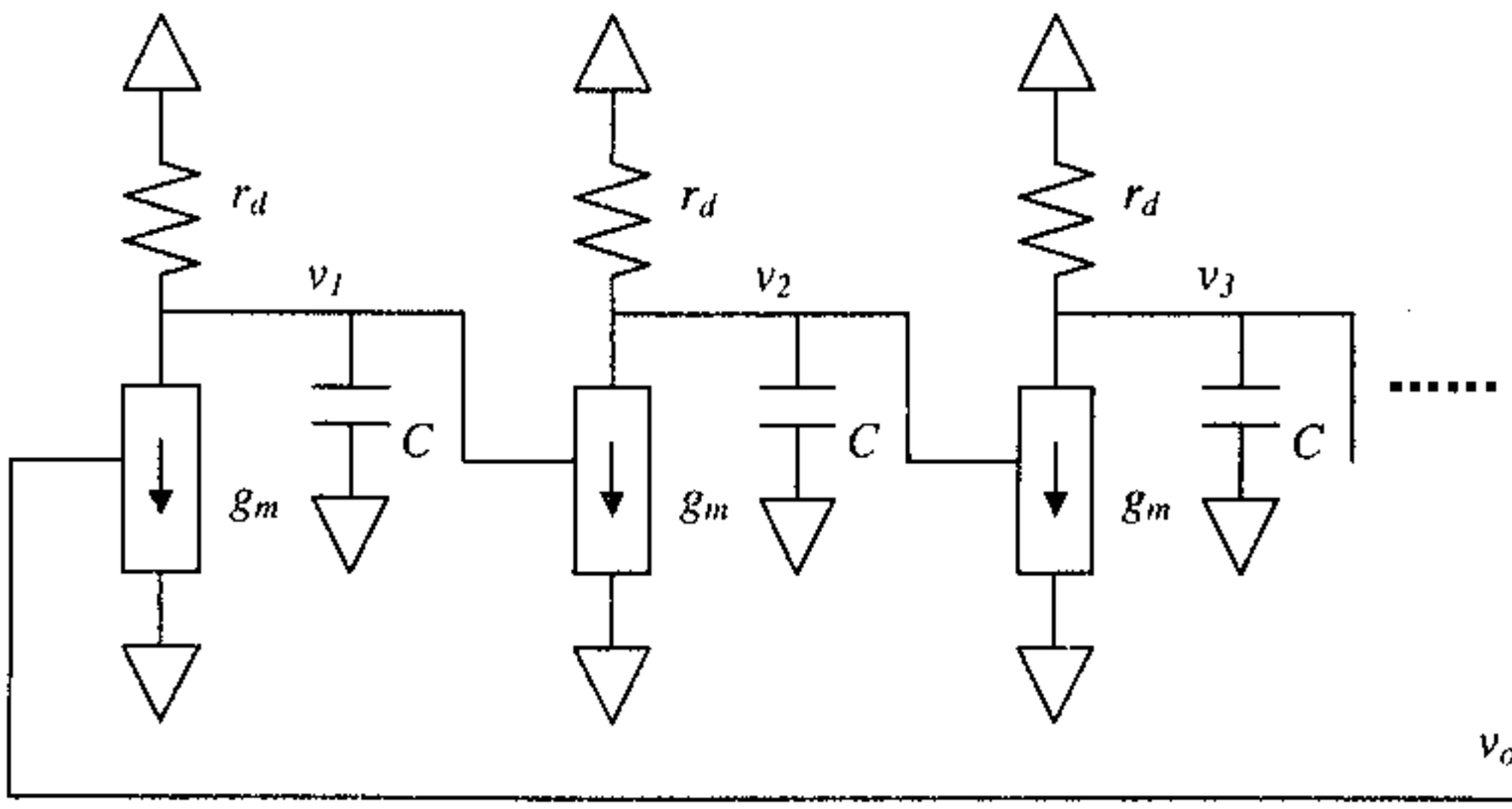
10. A ring oscillator shown in Figure (a) is often used to monitor gate delays of a certain VLSI process. The ring oscillator is composed of a series of N inverter gates, where N is an odd number (i.e., $N = 1, 3, 5, \ldots$). Because the output voltage of the series of inverters feedbacks to the input with an opposite sign, the loop will sustain oscillation of a fixed frequency. To calculate this frequency, we simplify the series of inverters as in Figure (b), where each inverter is modelled as a voltage controlled current source (g_m) and a resistor (r_d). And each inverter "sees" an output load of C.

    (a) (10%) Please setup a system of ordinary differential equations for the inverter output voltages ($v_1, v_2, v_3, \ldots$, and v_o) in Figure (b). (Hint: The current passing through the voltage controlled current source (g_m) is $g_m v_{input}$, if the controlling voltage is v_{input}. For example, the current source of the first inverter produces a current of $g_m v_o$.)

    (b) (10%) Please calculate the frequency of an N stage ring oscillator. (Hint: Laplace-transform the ODEs into the s domain and set $s = j\omega$. Solve for ω.)



(a) A ring oscillator.



(b) A simplified model of ring oscillator.