1. A coin with the probability of head $P\{H\} = p = 1-q$ is tossed n times.

   (a) (5 %) Find the probability that k heads are observed up to the n-th tossing but not earlier.

   (b) (5 %) Show that the probability that the number of heads is even equals $0.5[1 + (q - p)^n]$.

2. The random variables X and Y are independent with exponential densities
$$f_X(x) = \alpha \exp(-\alpha x)U(x), \qquad f_Y(y) = \beta \exp(-\beta y)U(y),$$
   where $U(x)$ is the unit step function at $x = 0$.

   (a) (5 %) Find the density of the random variable $Z = 2X + Y$.

   (b) (5 %) Find the density of the random variable $W = X - Y$.

3. (10 %) Let X_1 and X_2 be two continuous random variables with joint probability density function $f(x_1, x_2) = \begin{cases} 4x_1 x_2, & 0 < x_1 < 1, 0 < x_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$. Find the joint probability density function of $Y_1 = X_1^2$ and $Y_2 = X_1 X_2$.

4. (10%) Find the number of roots of the equation $z^8 - 5z^5 + z^2 - 1 = 0$ of absolute value less than 1.

5. (10%) Evaluate the integral $\frac{1}{2\pi i} \oint_C \frac{ze^z}{(z-2)^3} dz$ given that 2 is inside the contour C.

6. The linear operator L defined by $L(p(x)) = p'(x) + p(0)$, where $p'(x)$ denotes the derivative of $p(x)$, maps P_3 into P_2, where P_n is the space of all polynomials of degree less than n.

   (a) (7%) Find the matrix representation of L with respect to the ordered bases $\{x^2,\ x,\ 1\}$ of P_3 and $\{1,\ 1-x\}$ of P_2.

   (b) (4%) Let $p(x) = x^2 + 2x - 3$ in P_3; find the coordinates of $L(p(x))$ with respect to the ordered basis $\{1,\ 1-x\}$.

   (c) (4%) Repeat (b) for $q(x) = 4x^2 + 2x$ in P_3.

7. Given the vector space $C[-1,\ 1]$ with inner product $\langle f,\ g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$ and norm $\|f\| = (\langle f,\ f \rangle)^{1/2}$, where $C[-1,\ 1]$ denotes the space of all real-valued functions that are defined and continuous on the closed interval $[-1,\ 1]$.

   (a) (4%) Compute $\langle 1,\ x \rangle$.

   (b) (4%) Compute $\|1\|$ and $\|x\|$.

   (c) (7%) Find the best least squares approximation to $x^{1/3}$ on $[-1,\ 1]$ by a linear function $l(x) = c_1 + c_2 x$, i.e., find the best linear function $l(x) = c_1 + c_2 x$ which minimizes $\|x^{1/3} - l(x)\|$.

8. For the given differential equation:

$$y'' + 4y' + 3y = 6; \quad y'(0) = 4, y(0) = 0$$

(5%)(a) Find the homogeneous solutions, a particular solution, and the complete solutions.

(5%)(b) Use Laplace transform to solve the problem in the s domain, and then convert the solution into its time domain expression.

9. Find the Fourier series of the given function $f(\theta)$ which is assumed to have the (10%) period 2π.