

1. Consider two electric dipoles \vec{p}_1 , \vec{p}_2 , and a charge q , lying at positions \vec{r}_1 , \vec{r}_2 , and \vec{r}_q , respectively. Which of the following statements are true? (複選) (多勾者倒扣, 扣完為止) (15%)

- (a) The electric potential energy of the dipole \vec{p}_1 is $(\vec{p}_1 \cdot \vec{E}_1)$, where \vec{E}_1 is the electric field at \vec{r}_1 .
- (b) The electric field at \vec{r}_1 is $\vec{E}_1 = -\nabla_{\vec{r}_1}(V_1)$, where $V_1(\vec{r}_1) = q/|\vec{r} - \vec{r}_1| + V_{12}$, V_{12} being the electric potential at \vec{r}_1 due to the dipole \vec{p}_2 .
- (c) The potential energy between q and \vec{p}_2 is $V_{q2} = q[\vec{p}_2 \cdot (\vec{r}_2 - \vec{r}_q)]/|\vec{r}_2 - \vec{r}_q|^3$.
- (d) The electric potential at q due to dipole \vec{p}_2 is $-\vec{p}_2 \cdot (\vec{r}_2 - \vec{r}_q)/|\vec{r}_2 - \vec{r}_q|^3$.
- (e) The electric potential at \vec{r}_1 due to the dipole \vec{p}_2 is $V_{12} = -\vec{p}_2 \cdot (\vec{r}_1 - \vec{r}_2)/|\vec{r}_1 - \vec{r}_2|^3$.
- (f) None of the above is true.

2. Consider the following magnetic field

$$\vec{B}(x, y, z) = \begin{cases} (0, B_0, 0), & 0 \leq z \leq L \\ 0, & \text{elsewhere} \end{cases}, \text{ where } B_0 \text{ is a constant. Let } \vec{A}(x, y, z) \text{ be the corresponding vector}$$

field. Which of the following statements are true? (複選) (多勾者倒扣, 扣完為止) (10%)

(a) There are many choices for $\vec{A}(x, y, z)$ as long as they satisfy $\vec{B}(x, y, z) = \nabla \times \vec{A}(x, y, z)$.

(b) We can choose $\vec{A}(x, y, z) = 0$ for $|z| > L$.

(c) A possible choice for $\vec{A}(x, y, z)$ is the following.

$$\vec{A}(x, y, z) = \begin{cases} 0, & z < 0 \\ (B_0 z, 0, 0), & 0 \leq z \leq L \\ 0, & z > L \end{cases}.$$

(d) A possible choice for $\vec{A}(x, y, z)$ is the following.

$$\vec{A}(x, y, z) = \begin{cases} 0, & z < 0 \\ (B_0 z, 0, 0), & 0 \leq z \leq L \\ (B_0 L, 0, 0), & z > L \end{cases}.$$

(e) None of the above is true.

3.

- (a) A very long and very thin straight wire located along the z -axis carries a current I in the z -direction. Find the magnetic field intensity at any point in free space using Ampere's law in integral form. (5%)
- (b) Does the equation of continuity imply charge conservation? How? (5%)
- (c) Find electric field intensity due to an isolated point charge q . Starting from the differential form of the Gauss's law. (5%)
- (d) Explain the working principle of a lightning arrestor (避雷針). (5%)
- (e) Find the inductance per unit length of a very long solenoid having n turns per unit length. The permeability of the core is μ . The magnetic flux density of the solenoid is μnI . (5%)

4. The electric field intensity of a time-harmonic plane electromagnetic wave is given by

$\vec{E}(x, y, z, t) = \hat{a}_x 126 \times \sin[10^9 t - 6y + 8z - 0.2]$ Volt/m, where all the variables are in MKSA units. This wave propagates in a simple, nonmagnetic medium. Find numerically, including units if applicable, for

- (a) the unit vector pointing to the wavefront propagation direction, (Indicate the direction of the vector by plotting it in the y - z plane). (3%)
- (b) the wavelength along the wavefront normal direction, (3%)
- (c) the wavefront velocity or the phase velocity of the electromagnetic wave in the wavefront propagation direction, (3%)
- (d) the phase velocity in the z direction, (3%)
- (e) the wavelength in the y direction, (3%)
- (f) the refractive index of the medium, (3%)
- (g) the wave impedance, (3%)
- (h) the time-average intensity (power per unit area) of the wave. (4%)

Hint: the vacuum permittivity is $\epsilon_0 = \frac{1}{36\pi} \times 10^{-9}$ F/m.

and the vacuum permeability is $\mu_0 = 4\pi \times 10^{-7}$ H/m

5. Consider a short dipole antenna of length ℓ carrying a current $I_0 \cos(\omega t)$ located at origin and oriented along the z axis in free space. The electric field component of the wave at distance r very much greater than the wavelength is approximately given by

$$\vec{E}(R, \theta) = \hat{a}_\theta j \frac{30 \beta I_0 \ell}{R} \sin \theta e^{-j\beta R} \quad \text{where } \beta = \omega \sqrt{\mu_0 \epsilon_0}$$

- (a) Find the corresponding magnetic field $\vec{H}(R, \theta)$ (5%)
- (b) Find the time averaged poynting vector \vec{P}_{av} (5%)
- (c) Find the total power radiated by the dipole source by integrating the poynting vector over a spherical surface centered at the dipole. (5%)

(d) Show that the radiation resistance of this short dipole can be expressed as

$$R_{rad} = 80\pi^2 \left(\frac{\ell}{\lambda} \right)^2 \quad \text{where } \lambda \text{ is the wavelength.}$$

(5%)

(e) What is the total power radiate from a short antenna of $\ell = 0.01\lambda$ excited with 1 A current. (5%)

Note that the wave impedance is $120\pi\Omega$ here.

Cylindrical Coordinates (r, ϕ, z)

$$\nabla V = \mathbf{a}_r \frac{\partial V}{\partial r} + \mathbf{a}_\phi \frac{\partial V}{r \partial \phi} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (rA_r) + \frac{\partial A_\phi}{r \partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r} \begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\phi r & \mathbf{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \mathbf{a}_r \left(\frac{\partial A_z}{r \partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \mathbf{a}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \mathbf{a}_z \frac{1}{r} \left[\frac{\partial}{\partial r} (rA_\phi) - \frac{\partial A_r}{\partial \phi} \right]$$

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Spherical Coordinates (R, θ, ϕ)

$$\nabla V = \mathbf{a}_R \frac{\partial V}{\partial R} + \mathbf{a}_\theta \frac{\partial V}{R \partial \theta} + \mathbf{a}_\phi \frac{1}{R \sin \theta} \frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 A_R) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{R \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2 \sin \theta} \begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta R & \mathbf{a}_\phi R \sin \theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R \sin \theta) A_\phi \end{vmatrix} = \mathbf{a}_R \frac{1}{R \sin \theta} \left[\frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right] + \mathbf{a}_\theta \frac{1}{R} \left[\frac{1}{\sin \theta} \frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R} (RA_\phi) \right] + \mathbf{a}_\phi \frac{1}{R} \left[\frac{\partial}{\partial R} (RA_\theta) - \frac{\partial A_R}{\partial \theta} \right]$$

$$\nabla^2 V = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}$$