1.(a) Solve the second-order ordinary differential equation

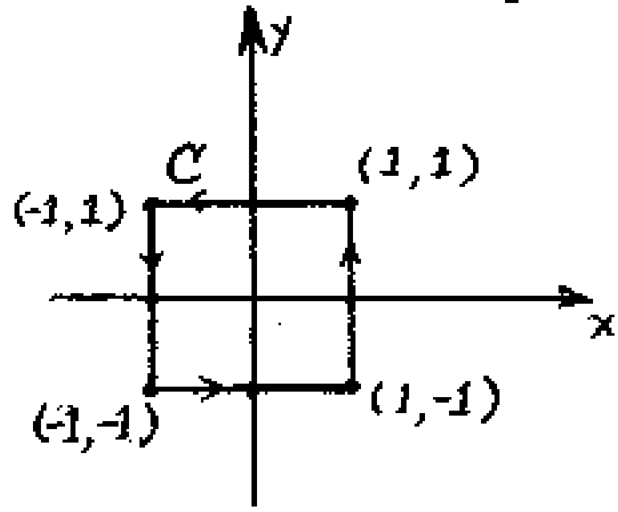
$$Y'' + 2Y' + Y = 1. \qquad (6\%)$$

(b) Solve the following equation

$(Y'')^2 + 4Y' \cdot Y'' + 2Y \cdot Y'' + 4(Y')^2 + 4Y \cdot Y' + Y^2 = 0$, where Y is a function of x, the prime and double prime denote, respectively, the first and second derivatives with respect to x. (14%)

[Hint: You may re-formulate the equation into a simple form.]

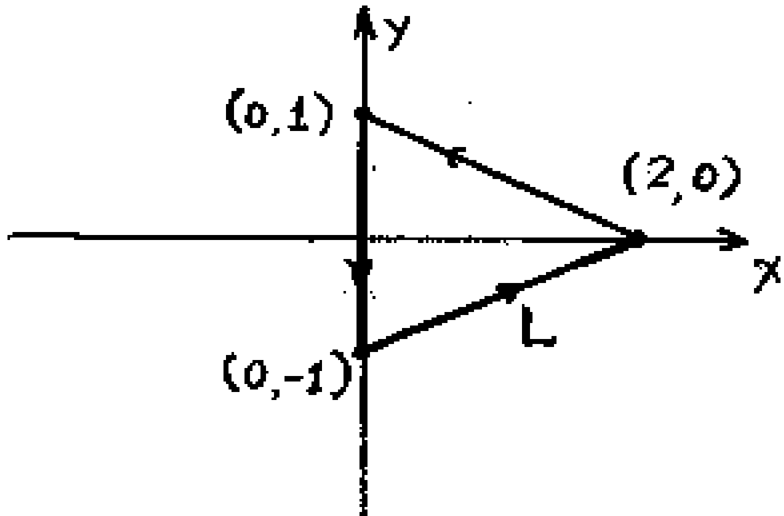
2. (a) Compute the complex integral $\oint_C \frac{\exp(Z)}{\exp(Z)-2}dZ$, where C is the counterclockwise contour (loop) as shown below. (10%)



(b) Compute the integral

$$\oint_L \frac{\exp(Z)-2}{\exp(Z)}dZ$$

where L is a counterclockwise loop path as shown below. (10%)



3. Suppose a laterally insulated long thin bar with length L and of constant cross section and homogeneous material is oriented along x-axis. The temperature $u(x,t)$ of the bar satisfies the following 1-D heat equation:

$u_t(x,t) = c^2 u_{xx}(x,t)$.

Find the temperature of the bar for any time $t > 0$ if the ends of the bar are kept at different constant temperatures $u(0,t) = U_1$ and $u(L,t) = U_2$ and initially $u(x, 0) = f(x)$. (10%)

4. (a) If $v(t)$ is one-sided that $v(t) = 0$ for $t < 0$ and $v(t) = V_0$ for $0 < t < a$, and 0 if $a < t < 2a$, and $v(t + 2a) = v(t)$ for $t > 0$, Find the Laplace transform of $v(t)$. (6%)

(b) If $y(t)$ satisfies the differential equation: $y' + y = v(t)$, and $y = 0$ for $t \leq 0$, find the steady-state solution of y by Laplace transform method. (7%)

5. The current $i(t)$of a circuit obeys the following differential equation:

$$d^2i(t)/dt^2 + di(t)/dt + i(t) = e_1(t)$$

where $e_1(t) = E \sin \omega t$ if $0 < t < p$ and 0 if $-p < t < 0$, $p = \pi/\omega$, and $e_1(t+2p) = e_1(t) \quad \forall t$.

(a)Find the complex Fourier series representation of $e_1(t)$. (6%)

(b)Find the steady-state current $i(t)$ for the circuit. (6%)

6. Prove the following two statements:

(a) Let $g_1, \ldots, g_n$ be the eigenvalues of matrix **A** then $1/g_1, \ldots 1/g_n$ are the eigenvalues of matrix $\mathbf{A}^{-1}$. (6%)

(b)If **x** is an eigenvector of **A** corresponding to an eigenvalue g, show that $\mathbf{y}=\mathbf{T}^{-1}\mathbf{x}$ is an eigenvector of $\mathbf{A}^{\wedge}= \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ corresponding to the same eigenvalue g. (7%)

7. Let the position vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, where $\vec{i}, \vec{j}$ and $\vec{k}$ are unit vectors along X-, Y-, and Z- coordinate axis. Let

$F(\vec{r}) = |\vec{r}|$, $\qquad \vec{V}(\vec{r}) = \vec{r} / |\vec{r}|^3$, $\qquad \vec{W}(\vec{r}) = \frac{1}{2}\vec{k} \times \vec{r}$.

S = a closed surface of arbitrary shape that encloses the origin,
C = the perimeter (周邊) of a square with area = 1, lying on the XY plane.

Which of the following statements are true? (複選) (12%)

a) $|\nabla F(\vec{r})| = 1$.

b) The surface integral $\oint_S \vec{V}(\vec{r}) \cdot d\vec{S} = 4\pi$.

c) $\nabla \times \vec{W}(\vec{r}) = 0$.

d) The line integral $\oint_C \vec{W}(\vec{r}) \cdot d\vec{l} = 1$.

e) All of the above are true.