1. Given measurement data as follows:

x	-2	-1	0	1
y	-3	-2	1	7

(a) (5%) Find the best least square fit to a linear function $y = ax + b$.

(b) (5%) Find the best least square fit to a quadratic function $y = ax^2 + bx + c$.

2. Determine $\lim_{m \to \infty} A^m$ for each of the following matrices, if it exists.

(a) (5%) $A = \begin{pmatrix} 2 & -0.5 & -1 \\ 1 & 0.5 & -1 \\ 1 & -0.5 & 0 \end{pmatrix}$, (b) (5%) $B = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 2 & 0 \\ 2 & -1 & -1 \end{pmatrix}$.

3. (10%) For each of the following statement, whether it is correct? (Give your answer simply in terms of "Y" (for Yes) or "N" (for No) and you don't need to give reasons.)

(a) X and Y are two random variables. It is generally true that
$E[(X - E(X))(Y - E(Y))]^2 \le E[(X - E(X))^2]E[(Y - E(Y))^2]$.

(b) Two random variables X and Y are statistically independent if and only if they are uncorrelated.

(c) If X is a discrete Poisson random variable with parameter λ where $\lambda > 0$, it is generally true that $E[X] = \lambda$ and $Var(X) = \lambda^2$.

(d) If X and Y are two statistically independent continuous random variables, then $E[Y \mid X] = E[Y]$.

(e) $X_1, X_2, \ldots, X_N$ are N mutually independent random variables. Each of them has the same mean μ and the same variance σ^2. If $Y = \frac{1}{N}\sum_{i=1}^{N} X_i$, the mean of Y is also μ and the variance of Y is also σ^2.

4. X and Y are two statistically independent continuous random variables. X is uniformly distributed in [0, 20] and Y is uniformly distributed in [50, 80].

(a) (5%) What is the probability density function of $X + Y$.

(b) (5%) What is the smallest integer K such that $P(X + Y \ge K) \le \frac{1}{30}$?

5. Suppose that the joint probability mass function $p_{X,Y,Z}(x,y,z)$ of three discrete type random variables X, Y and Z taking values in $\{0,1,2\}$ is

$$\begin{array}{lll} p_{X,Y,Z}(0,0,1)=\frac{1}{8}, & p_{X,Y,Z}(0,1,1)=\frac{1}{16}, & p_{X,Y,Z}(0,1,2)=\frac{1}{8}, \\ p_{X,Y,Z}(1,0,0)=\frac{1}{16}, & p_{X,Y,Z}(1,0,1)=\frac{1}{8}, & p_{X,Y,Z}(1,1,1)=\frac{1}{16}, \\ p_{X,Y,Z}(1,2,0)=\frac{1}{8}, & p_{X,Y,Z}(1,2,2)=\frac{1}{32}, & p_{X,Y,Z}(2,0,0)=\frac{1}{8}, \\ p_{X,Y,Z}(2,0,1)=\frac{1}{16}, & p_{X,Y,Z}(2,2,1)=\frac{1}{32}, & p_{X,Y,Z}(2,2,2)=\frac{1}{16} \end{array}$$

and zeros for the rest 15 configurations of X, Y, Z.

(a) (5%) Are X and Z statistically independent? Why?

(b) (5%) What is the conditional expectation $E[X|Y=y]$ of X on Y?

6. Consider the following 3×5 matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & -1 & -3 \\ 1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 1 & -2 & 0 \end{bmatrix}.$$

(a) (5%) Find a basis for the nullspace $\text{Null}(A) = \{\boldsymbol{x} \in R^5 | A\boldsymbol{x} = 0\}$ of A.

(b) (5%) Find a basis for the column space $\text{Col}(A) = \{A\boldsymbol{y} | \boldsymbol{y} \in R^5\}$ of A.

7. (6%) Which of the following statements are ture? (Proofs are not needed. Simply choose the true statements. No partial credit for this problem)

(a) The problem

$$y'(x) = 1 + y^2(x), \quad y(0) = 0$$

has a unique solution for all x in $[0,1]$.

(b) The problem

$$|y'(x)| + |y(x)| = 0, \quad y(0) = 1$$

has a unique solution for all x in $[0,1]$.

(c) The problem

$$y'(x) = \sqrt{|y(x)|}, \quad y(0) = 0$$

has a unique solution for all x in $[0,1]$.

(d) The problem

$$y''(x) + y(x) = 0, \quad y(0) = 0, y(\pi) = 0$$

has a unique solution for all x in $[0,\pi]$.

(e) The problem

$$y''(x) + 4y(x) = 8x^2, \quad y(0) = 0$$

has a unique solution for all x in $[0,1]$.

8. (7%) Consider the following homogeneous linear systems in two equations with constant coefficients:

$$y_1' = a_{11}y_1 + a_{12}y_2,$$
$$y_2' = a_{21}y_1 + a_{22}y_2.$$

Let $p = a_{11} + a_{22}$, $q = a_{11}a_{22} - a_{12}a_{21}$ and $\Delta = p^2 - 4q$. Also, let P_0 be the critical point when $y_1 = y_2 = 0$. Which of the following statements are ture? (Proofs are not needed. Simply choose the true statements. No partial credit for this problem)

(a) P_0 is a saddle point if $q < 0$.
(b) P_0 is a spirial point if $p \neq 0$ and $\Delta > 0$.
(c) P_0 is a center if $p = 0$ and $q \neq 0$.
(d) P_0 is unstable if $p > 0$ and $q > 0$.
(e) P_0 is stable and attractive if $p < 0$ and $q > 0$.

9. (7%) Please solve the following integral equation:

$$y(t) = \sin t + \int_0^t y(\tau)\sin(t-\tau)d\tau.$$

10. (10%) Let z_1, z_2 and z be complex numbers and $i = \sqrt{-1}$. Please show that

(a) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.
(b) $sinh(iz_1) = i\sin(z_1)$.
(c) $cosh(iz_1) = \cos(z_1)$.
(d) $sinh(z_1 + z_2) = sinh(z_1)cosh(z_2) + cosh(z_1)sinh(z_2)$.
(e) $sinh(z) = sinh(x)cos(y) + icosh(x)sin(y)$, where $z = x + iy$ with x, y real.

Note that $sinh(z) = \frac{e^z - e^{-z}}{2}$ and $cosh(z) = \frac{e^z + e^{-z}}{2}$.

11. (10%) Find ϕ (in degrees) so that the following periodic waveform does not contain any 5^{th} harmonic voltage.