1. Consider a $n \times n$ real-valued matrix A. Which of the following statements are equivalent to "A is nonsingular"? (10%) (Proofs are not needed. Simply choose the equivalent statements. No partial credit for this problem.)

   (a) A is invertible.

   (b) $A\mathbf{x} = \mathbf{0}$ has a solution $\mathbf{0}$.

   (c) The system of n linear equations in n unknowns $A\mathbf{x} = \mathbf{e}_1$ has a unique solution, where $\mathbf{e}_1 = (1, 0, \ldots, 0)^T$.

   (d) $A^2 + 3A + I$ is nonsingular.

   (e) $A^2 + 4A$ is nonsingular.

   (f) The column vectors of A are linearly independent.

   (g) The row vectors of A spans $\mathcal{R}^n$.

   (h) A is similar to some matrix C.

   (i) A is a transition matrix with respect to some ordered basis to the standard basis.

   (j) A is a matrix representing some linear transformation.

2. Let

$$A = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{6} \end{pmatrix}.$$

   (a) Find $\lim_{n \to +\infty} A^n$. (8%)

   (b) For any nonzero vector $\mathbf{x} \in \mathcal{R}^2$, define

$$\rho(x) = \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

   Find the minimum of $\rho(\mathbf{x})$ over the set of nonzero vectors in $\mathcal{R}^2$. (7%)

3. Solve the following differential equations:

(a) $(y\sin y - 3xy)y' = y$. (8%)

(b) If $Y(t) = [y_1(t)\ y_2(t) \ldots y_n(t)]$ is known where yi's(t) are the n linearly independent solutions to the homogeneous linear system : $y'(t) = A(t)y(t)$, where $A(t)$ is an n x n time-varying matrix, find the general solution to the forced linear system $y'(t) = A(t)y(t) + g(t)$. (7%)

4. For a series RLC circuit, the differential equation for the inductor current is $i''(t) + i'(t) + i(t) = e(t)$. Solve the following sub-problems:

(a) If $e(t)=1-e^{-t}$ for $0<t<1$ and 0 else, and zero initial values: $i'(t) = 0$, $i(t) = 0$ at $t = 0$, find the inductor current for $t \geq 0$. (5%)

(b) Repeat (a) for $t \geq 1$ if $e(t) = 2-2e^{-(t-1)}$ for $1<t<3$ and $-1+e^{-(t-3)}$ for $3<t<4$ and 0 elsewhere and the circuit has zero initial values. (5%)

(c) Find the inductor current if $e(t) = 1$ for $0<t<1$ and 0 for $1<t<2$, $e(t+2) = e(t)$ for $t > 0$. and zero initial values.(5%)

(d) Find the steady state inductor current in (c) by using Fourier series method. (5%)

5. (a) Express $\cos(\frac{z}{z-1})$ into a power series for some region of the complex plane. (5%)

(b) Compute the following complex integrals

$$\int_{C1} \cos(\frac{z}{z-1})dz \quad \text{and} \quad \int_{C2} \cos(\frac{z}{z-1})dz$$

where C1 and C2 are counterclockwise contours (circles) with center z=0 and radii of 0.5 and 2,respectively. (10%)

6. Justify the following statements by providing a proof for each correct statement or a rectification for each incorrect statement: (3% each)

(a) X and Y are marginally normal if and only if they are jointly normal.

(b) If Z_1 and Z_2 are independent standard normal random variables, then $Y = Z_1^2 + Z_2^2$ is also a normal random variable.

(c) If $X_1, X_2, \ldots, X_n$ are pairwise independent, then

$$Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var(X_i)$$

(d) The sample variance of *i.i.d.* random variables $X_1, X_2, \ldots, X_n$ is given by $\sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n}$, where $\overline{X}$ is sample mean.

(e) If $Var(X) = 0$, then $X = E[X]$ with probability 1.

7. Suppose that the joint density of X ad Y is given by

$$f(x,y) = \begin{cases} \dfrac{e^{-x/y}\, e^{-y}}{y}, & 0 < x < \infty, 0 < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

Find (a) $P\{X > 1 \mid Y = y\}$. (5%)

(b) $E[X \mid Y = y]$. (5%)