國立清華大學命題紙

八十七學年度 電機工程 系（所） 乙 組碩士班研究生入學考試

科目 工程數學 科號 3001 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. Write down a matrix with the required property or explain why no such matrix exists.

   (a) (3%) Column space has basis $\{(1,2,-1)^T\}$, and row space has basis $\{(2,1)\}$.

   (b) (3%) Column space has basis $\{(1,1,1)^T\}$, and nullspace has basis $\{(1,0,-1)^T\}$.

   (c) (4%) The vector $(1,2,-2)^T$ is in the nullspace and $(-2,1,0)$ is in the row space and the determinant $=1$.

2. Let T be the linear transformation from $\mathcal{R}^2$ into $\mathcal{R}^2$, where $\mathcal{R}$ is the set of all real numbers, defined by

$$\boldsymbol{T}(x_1,x_2)=(3x_1/2+x_2/2,x_1/2+3x_2/2).$$

   (a) (10%) Define $\boldsymbol{T}^n$ recursively by $\boldsymbol{T}^n(x_1,x_2)=\boldsymbol{T}(\boldsymbol{T}^{n-1}(x_1,x_2))$ for $n\geq 2$ and $\boldsymbol{T}^1(x_1,x_2)=\boldsymbol{T}(x_1,x_2)$. Give a rule for $\boldsymbol{T}^n$ like the one which defines $\boldsymbol{T}$ for every n.

   (b) (5%) Find an orthonormal basis in which the matrix representation of $\boldsymbol{T}$ is diagonal.

3. (5%) Continue from Problem 2. Suppose that X_1 and X_2 are independent and identically distributed (i.i.d.) Bernoulli random variables with parameter 0.5, i.e., $P(X_1=1)=P(X_1=0)=P(X_2=1)=P(X_2=0)=0.5$. Find the probability $P(T(X_1,X_2)=(2,2))$.

4. (5%) A matrix $\boldsymbol{A}$ is called skew-symmetric if $\boldsymbol{A}^T=-\boldsymbol{A}$. Show that for a real and skew-symmetric matrix, all of its eigenvalues are purely imaginary.

5. (10%) Consider a nonnegative integer valued random variable X. Suppose that $E[X]=10$ and $P(X=0)=0.1$. Find $E[\max(0,X-1)]$.

6. (10%) Let $\{Y_i, i=1,2,\ldots\}$ be a sequence of independent and identically distributed (i.i.d.) Bernoulli random variables with parameter p, i.e., $P(Y_1=1)=p$ and $P(Y_1=0)=1-p$. Also, let N be a Poisson random variable with parameter λ, i.e.,

$$P(N=k)=\frac{e^{-\lambda}\lambda^k}{k!}, k=0,1,2,\ldots.$$

   Consider the random variable $Z=\sum_{i=1}^{N}Y_i$. Assume that $\{Y_i, i=1,2,\ldots\}$ is independent of N. Find the probability $P(Z=k)$.

7. (a) (5%) Show that the linear fractional transform:

$$\omega = \frac{z - z_0}{cz - 1}, c = \bar{z}_0, |z_0| < 1$$

maps the unit disk in the z-plane onto the unit disk in the ω-plane.

(b) (5%) Find the potential between the infinite long cylinders $C_1 : |z| = 1$ (grounded, i.e., $U_1 = 0$) and $C_2 : |z - 2/5| = 2/5$ (having potential $U_2 = 100$ volts).

8. (10%) Integrate the following function around the contour C: $f(z) = [\ln(z+3) + \cos z]/(z+1)^2$, C: the boundary of the square with vertices $2, -2, 2i, -2i$, counterclockwise.

9. Let $G(\omega)$ be the Fourier Transform of a real function $g(x)$, i.e.,

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-j\omega x} dx.$$

Now a real function $f(x)$ is related to $g(x)$ as follows:

$$f(x) = \frac{2}{3} g(x) + \frac{1}{4}[g(x-a) + g(x+a)],$$

where a is a positive real number. Let $F(\omega)$ be the Fourier transform of $f(x)$. Answer the following questions:

(a) (8%) What is the relationship between $F(\omega)$ and $G(\omega)$?

(b) (7%) If $f(x)$ is known, can one obtain $g(x)$ accordingly? If the answer is yes, how?

10. (10%) Solve the following initial value problem by means of Laplace Transform.

$$\begin{cases} y_1' = 6y_1 + 9y_2, & y_1(0) = -3, \quad y_2(0) = -3. \\ y_2' = y_1 + 6y_2. \end{cases}$$