國 立 清 華 大 學 命 題 紙

八十七學年度 電機工程 系（所） 甲 組碩士班研究生入學考試

科目 工程數學 科號 2901 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. Write down a matrix with the required property or explain why no such matrix exists.

   (a) (3%) Column space has basis $\{(1,2,-1)^T\}$, and row space has basis $\{(2,1)\}$.

   (b) (3%) Column space has basis $\{(1,1,1)^T\}$, and nullspace has basis $\{(1,0,-1)^T\}$.

   (c) (4%) The vector $(1,2,-2)^T$ is in the nullspace and $(-2,1,0)$ is in the row space and the determinant $= 1$.

2. Let T be the linear transformation from $\mathcal{R}^2$ into $\mathcal{R}^2$, where $\mathcal{R}$ is the set of all real numbers, defined by

$$T(x_1,x_2) = (3x_1/2 + x_2/2, x_1/2 + 3x_2/2).$$

   (a) (10%) Define T^n recursively by $T^n(x_1,x_2) = T(T^{n-1}(x_1,x_2))$ for $n \geq 2$ and $T^1(x_1,x_2) = T(x_1,x_2)$. Give a rule for T^n like the one which defines T for every n.

   (b) (5%) Find an orthonormal basis in which the matrix representation of T is diagonal.

3. (5%) Continue from Problem 2. Suppose that X_1 and X_2 are independent and identically distributed (i.i.d.) Bernoulli random variables with parameter 0.5, i.e., $P(X_1 = 1) = P(X_1 = 0) = P(X_2 = 1) = P(X_2 = 0) = 0.5$. Find the probability $P(T(X_1,X_2) = (2,2))$.

4. Are the following sets of vectors $a = (a_1, a_2, \ldots, a_n)$ in $\mathcal{R}^n$, where $\mathcal{R}$ is the set of all real numbers, are subspaces of $\mathcal{R}^n$ $(n \geq 3)$? If yes, find its dimension. If no, explain why.

   (a) (5%) All a such that $a_1 \geq 0$.

   (b) (5%) All a such that $a_1 + a_2 + 2a_3 = 0$, $3a_1 - a_2 + a_3 = 0$, and $3a_1 - 5a_2 - 4a_3 = 0$.

5. (a) (5%) Show that the linear fractional transform:

$$\omega = \frac{z - z_0}{cz - 1}, c = \bar{z}_0, |z_0| < 1$$

   maps the unit disk in the z-plane onto the unit disk in the ω-plane.

   (b) (5%) Find the potential between the infinite long cylinders $C_1 : |z| = 1$ (grounded, i.e., $U_1 = 0$) and $C_2 : |z - 2/5| = 2/5$ (having potential $U_2 = 100$ volts).

6. (10%) Integrate the following function around the contour C: $f(z) = [\ln(z+3) + \cos z]/(z+1)^2$, C: the boundary of the square with vertices $2, -2, 2i, -2i$, counterclockwise.

7. Let $G(\omega)$ be the Fourier Transform of a real function $g(x)$, i.e.,

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-j\omega x} dx.$$

Now a real function $f(x)$ is related to $g(x)$ as follows:

$$f(x) = \frac{2}{3} g(x) + \frac{1}{4}[g(x-a) + g(x+a)],$$

where a is a positive real number. Let $F(\omega)$ be the Fourier transform of $f(x)$. Answer the following questions:

(a) (8%) What is the relationship between $F(\omega)$ and $G(\omega)$?

(b) (7%) If $f(x)$ is known, can one obtain $g(x)$ accordingly? If the answer is yes, how?

8. (a) (5%) Solve the initial value problem,

$$y'' - 4y' + 3y = 0, \quad y(0) = -1, \quad y'(0) = 3.$$

(b) (10%) Solve the initial value problem,

$$y'' - 4y' + 3y = 4e^{3x}, \quad y(0) = -1, \quad y'(0) = 3.$$

9. (10%) Solve the following initial value problem by means of Laplace Transform.

$$\begin{cases} y_1' = 6y_1 + 9y_2, & y_1(0) = -3, \quad y_2(0) = -3. \\ y_2' = y_1 + 6y_2. & \end{cases}$$