八十六學年度 電機工程 系(所) 乙 組碩士班研究生入學考試

科目 工程數學 科號 3101 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. A box contains n black balls. At each stage a black ball is removed and a new ball, that is black with probability p and white with probability (1-p), is put into the box. Please find the expected number of stages needed until there are no more black balls in the box. (10%)

2. Denote the owl and wood rat populations at time k by $X_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$, where k is the time in months, O_k is the number of owls in the region studied, and R_k is the number of rats (measured in thousands). Suppose that $\begin{cases} O_{k+1} = 0.5 \cdot O_k + 0.4\, R_k \\ R_{k+1} = -P \cdot O_k + 1.1\, R_k \end{cases}$ where p is a positive parameter to be specified. Please find the value of p that can let the system enter the steady state (i.e., $\lim_{n \to \infty}(X_{n+1}) = X_n$).

(10%)

3. a. Suppose that f(x) and g(x) are piecewise continuous, bounded, and absolutely integral on x-axis. Based on the definition of fourier transform, please prove the following convolution theorem. (10%)

$$F(f * g) = \sqrt{2\pi}\, F(f) F(g)$$

where * denotes the convolution operation and F denotes the fourier transform.

b. Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. Furthermore $f'(x)$ be absolute integral on x-axis. Based on the definition of fourier transform, please prove

(i) $F\{f'(x)\} = i\omega F\{f(x)\}$, $i \triangleq \sqrt{-1}$

(ii) $F\{f''(x)\} = -\omega^2 \{f(x)\}$ (10%)

4. (10%) The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{y} e^{-x/y} e^{-y} & \text{if } 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find $E[X\,Y]$.

5. (10%) Show that if a matrix U is both unitary and Hermitian, then any eigenvalues of U must either equal 1 or -1.

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6. In the complex plane, consider the function $f(z) = (z+1)^2$ and the region D which is the triangular region with vertices at the points $z = 0$, $z = 2$, and $z = i$. Find points in D where $|f(z)|$ has its maximum and minimum values. (10%)

7. Consider the transformation $w = u + iv = z^2$, where $z = x + iy$.

   (a) Let C be the curve $x = 1$ and the positive orientation of C be the direction of increasing y. Find and sketch the image Γ of C under the above transformation. Also indicate the corresponding positive orientation of Γ. (5%)

   (b) What is the resulting angle of rotation from C to Γ at the point $z = 1 + i$? (5%)

8. Let E and F be two events in a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Please show the following two statements:

   (a) (5%) $Pr\{E \cap F\} \geq Pr\{E\} + Pr\{F\} - 1$.

   (b) (5%) If E and F are statistically independent, i.e. $Pr\{E \cap F\} = Pr\{E\} \cdot Pr\{F\}$, then their complements E^c and F^c are also statistically independent, i.e. $Pr\{E^c \cap F^c\} = Pr\{E^c\} \cdot Pr\{F^c\}$.

9. Let A be an $m \times n$ real matrix and A^t the transpose of A. Please show that

   (a) (5%) An n-tuple v is in the null space of the matrix A^tA, i.e. $(A^tA)v = 0$ if and only if v is in the null space of A, i.e. $Av = 0$.

   (b) (5%) The rank of matrix A^tA is equal to the rank of matrix A.