1. A simple harmonic oscillator satisfies the equation

$y''(t) + r\, y'(t) + \omega_0^2\, y(t) = f_0 \cos(\omega t)$,

where $y(t)$ is the displacement of the oscillator at time t, r the damping constant, ω_0 the natural frequency, $f_0 \cos(\omega t)$ the driving force. Suppose the system is in the regime of underdamping.

i) Derive the steady- state solution to the above equation. (5%)

ii) Derive the condition of resonance, i.e., when the steady-state solution has the maximum amplitude. (5%)

2. Find the general solution to the following wave equation

$\partial^2 f(x,t)/\partial^2 x - (1/c^2)\, \partial^2 f(x,t)/\partial^2 t = 0$,

subject to the condition that $f(x{=}0,t) = f(x{=}L, t) = 0$ for all t, where c is the wave velocity. (15%)

3. a) Find the Laplace transform ($\mathcal{L}$) of the function

$$f(t) = \begin{cases} 1 & \text{if} \quad 0 < t < \pi \\ 0 & \text{if} \quad \pi < t < 2\pi \\ \sin(t) & \text{if} \quad t > 2\pi \end{cases} \qquad (5\%)$$

b) Find the inverse transform ($\mathcal{L}^{-1}(s)$) of the function

$$\frac{1}{s^2(s-a)} \qquad (10\%)$$

4. Find the Fourier transform $F(s)$ of the functions

(a) $f(x) = e^{-|x|}$ (5%)

(b) $f(x) = a \cdot \pi\left\{\dfrac{x-b}{c}\right\}$, where $\pi(y) = \begin{cases} 1 & |y| < 1/2 \\ 0 & |y| > 1/2 \end{cases}$ (5%)

5. For an NxN matrix, denote its (i,j) element by A_{ij}. Define the trace of A by

$Tr(A) = \sum_{i=1}^{N} A_{ii}$. Prove that $Tr(A)$ is invariant under similarity transformation; that is,

$Tr(A) = Tr(U^{-1}AU)$, where U is an arbitrary non-singular matrix. (10%)

6. Let $A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$

Solve the differential equation $\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. (15%)

7. The integral $\int_{0}^{\infty} \frac{dx}{x^4 + (a+b)x^2 + ab}$ is found to equal $\frac{\pi}{4}$, where both

a and b are positive real numbers. If a and b are related by $ab = 1$,

then find a and b. (13 %)

8. If the integral $\displaystyle\oint_C \frac{(z-b)\,dz}{(z-a)}$ is equal to 0, find a and b.

Here C is a counterclockwise contour along the unit circle shown below, and both a and b are complex numbers. (12%)