國 立 清 華 大 學 命 題 紙

八十四學年度 電機工程 所 乙 組碩士班研究生入學考試

科目 工程數學 科號 2301 共 4 頁第 1 頁 *請在試卷【答案卷】內作答

Problem #1. 10%

(a) What is the inverse Laplace transform of $\frac{1 - e^{-as}}{s + b}$, where $a>0$ and $b>0$? (3%)

(b) Let $y^{(i)}(t)$ denote the ith derivative of $y(t)$. Consider the initial value problem

$$y(t) + \sum_{i=1}^{n} a_i y^{(i)}(t) = \gamma(t), \quad y^{(i)}(0) = 0,\ i=1, 2, \ldots, n-1, \text{ and } y(0) = 0.$$

Let the Laplace transforms of the solution $y(t)$ and $\gamma(t)$ be $Y(s)$ and $R(s)$, respectively. What is the relationship between $Y(s)$ and $R(s)$? (3%)

(c) Same as Part (b); if the solution $y(t)$ can be written as

$$y(t) = \int_0^t q(t-\tau)\gamma(\tau)\, d\tau,$$

what is $q(t)$? (4%)

Problem #2. 10%

Find the Fourier transform of the following function:

$$f(t) = \begin{cases} \sqrt{2\pi}, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Problem #3. 20%

Let $X_1, X_2, \ldots, X_{100}$ be independent and identically distributed (i.i.d.) Bernoulli random variables with parameter p, i.e., $P(X_1=1) = p$ and $P(X_1=0) = 1-p$. Let

$$Z = \frac{1}{100}\left(\sqrt{X_1} + \sqrt{X_2} + \ldots + \sqrt{X_{100}}\right)$$

(a) Find the mean of Z, i.e., E[Z].

(b) Find the variance of Z, i.e., $Var[Z] = E[(Z-E[Z])^2]$.

(c) Find the conditional expectation $E[X_1|Z]$.

(d) Find the conditional expectation $E[X_1X_2|Z]$.

Problem #4. 15%

(a) Let a function f(z) be analytic throughout a simply connected domain D and let z_0 be the only zero (with order m) of f(z) in D. Show that if C is a positively oriented (counterclockwise) simple closed contour in D that encloses z_0, then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = m \qquad (8\%)$$

where $f'(z) = df(z)/dz$.

(b) Use the result of Part(a) to prove the following property. Let D be a simply connected domain throughout which a function f is analytic and $f'(z) \neq 0$. Let C denote a simple closed contour in D, described in the positive sense, such that $f(z) \neq 0$ at any point on C. Then, if f has N zeros interior to C, that number is given by

$$N = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz \qquad (7\%)$$

Problem #5. 10%

Let D be the differentiation operator on the set P^2 of all polynomials over real numbers of degree ≤ 2, i.e., $D(a+bt+ct^2) = b + 2ct$. Let B be the ordered basis $\{1,(t+1),(t+1)^2\}$ of P^2.

(a) Please find the matrix representation $[D]_B$ of D relative to the basis B. (3%)

(b) Let B' be another ordered basis $\{t^2+t, t-1, t^2-t+1\}$ of P^2. Please find the matrix representation $[D]_{B'}$ of D relative to the basis B' and relate $[D]_B$ to $[D]_{B'}$. (7%)

Problem #6. 10%

True or false. You should give reasons or counterexamples, otherwise no credits.

(a) If A and B are two diagonalizable n×n matrices, then AB is also diagonalizable. (2%)

(b) All divergent sequences form a subspace of the vector space

$$R^{\infty} = \{(x_1, x_2, \ldots) \mid x_i \in R, \forall i \geq 1\}. \qquad (2\%)$$

(c) Let

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} \text{ and } b = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

Then, there are more than one least–squares solutions of the linear system

$$A\,x = b \qquad (3\%)$$

where x is a 2×1 column vector.

(d) There exists a 10×7 matrix A of rank 7 such that A has a left inverse but no right inverse? (3%)

Problem #7. 15%

Consider the following matrix and vector:

$$A = \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.2 \\ 0.3 & 0.1 & 0.7 \end{bmatrix} \text{ and } z = \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}.$$

(a) Is A diagonalizable? Why ? (4%)

(b) Find the eigenvalues of A^{-1}. (2%)

(c) Find the eigenvalues of A^3. (2%)

(d) Find $\lim_{k\to\infty} A^k z$. (7%)

Problem #8. 10%

Assume that X and Y are independent random variables with probability density function (p.d.f.) ce^{-cx}, $x\geq 0$ and ce^{-cy}, $y\geq 0$, respectively, where $c>0$. Let $Z=X-Y$. Please find the p.d.f., mean and variance of Z.