Problem #1. 15%

(a) Find the particular solution which satisfies the indicated condition:

$$\frac{dx(t)}{dt} = k\,x(t)\,(1-x(t));\ x(0) = \frac{1}{2}$$

(b) Discuss the steady–state behavior of x(t) as $t \to \infty$.

(c) What will happen to the solution x(t) if x(0) = 3/2?

Problem #2. 15%

Use the Laplace transform to solve the following variable–coefficient linear differential equation:

$$t\,\frac{d^2y(t)}{dt^2} + 2(t-1)\,\frac{dy(t)}{dt} + (t-2)\,y(t) = 0$$

Problem #3. 10%

Find the Fourier transform of the following function:

$$f(t) = \begin{cases} \sqrt{2\pi}, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Problem #4. 20%

Let X_1, X_2, ..., X_{100} be independent and identically distributed (i.i.d.) Bernoulli random variables with parameter p, i.e., $P(X_1=1) = p$ and $P(X_1=0) = 1-p$. Let

$$Z = \frac{1}{100}\left(\sqrt{X_1} + \sqrt{X_2} + \ldots + \sqrt{X_{100}}\right)$$

(a) Find the mean of Z, i.e., E[Z].

(b) Find the variance of Z, i.e., $\mathrm{Var}[Z] = E[(Z-E[Z])^2]$.

(c) Find the conditional expectation $E[X_1|Z]$.

(d) Find the conditional expectation $E[X_1X_2|Z]$.

Problem #5. 15%

(a) Let a function f(z) be analytic throughout a simply connected domain D and let z_0 be the only zero (with order m) of f(z) in D. Show that if C is a positively oriented (counterclockwise) simple closed contour in D that encloses z_0, then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = m \qquad (8\%)$$

where $f'(z) = df(z)/dz$.

(b) Use the result of Part(a) to prove the following property. Let D be a simply connected domain throughout which a function f is analytic and $f'(z) \neq 0$. Let C denote a simple closed contour in D, described in the positive sense, such that $f(z) \neq 0$ at any point on C. Then, if f has N zeros interior to C, that number is given by

$$N = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz \qquad (7\%)$$

Problem #6. 10%

True or false. You should give reasons or counterexamples, otherwise no credits.

(a) If A and B are two diagonalizable n×n matrices, then AB is also diagonalizable. (2%)

(b) All divergent sequences form a subspace of the vector space

$$\mathbb{R}^{\infty} = \{(x_1, x_2, \ldots) | x_i \in \mathbb{R}, \forall\, i \geq 1\}. \qquad (2\%)$$

(c) Let

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} \text{ and } b = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

Then, there are more than one least–squares solutions of the linear system

$$A\,x = b \qquad (3\%)$$

where x is a 2×1 column vector.

(d) There exists a 10×7 matrix A of rank 7 such that A has a left inverse but no right inverse? (3%)

Problem #7. 15%

Consider the following matrix and vector:

$$A = \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.2 \\ 0.3 & 0.1 & 0.7 \end{bmatrix} \text{ and } z = \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}.$$

(a) Is A diagonalizable? Why ? (4%)

(b) Find the eigenvalues of A^{-1}. (2%)

(c) Find the eigenvalues of A^3. (2%)

(d) Find $\lim_{k\to\infty} A^k z$. (7%)