類組：電機類　科目：工程數學C(3005)　

※請在答案卡內作答

- 本測驗試題為多選題（答案可能有一個或多個），請選出所有正確或最適當的答案，並請用2B鉛筆作答於答案卡。
- 共二十題，每題五分。每題ABCDE每一選項單獨計分；每一選項的個別分數為一分，答錯倒扣一分。

Notation: In the following questions, underlined letters such as $\underline{a}$, $\underline{b}$, etc. denote column vectors of proper length; boldface letters such as $\mathbf{A}, \mathbf{B}$, etc. denote matrices of proper size; $\mathbf{A}^\top$ means the transpose of matrix $\mathbf{A}$, and $\mathbf{A}^\dagger$ is the hermitian transpose (a.k.a. conjugate transpose) of $\mathbf{A}$. $\mathbf{I}_n$ is the $(n \times n)$ identity matrix. $\|\underline{a}\|$ means the Euclidean norm of vector $\underline{a}$. $\mathbb{R}$ is the usual set of all real numbers; $\mathbb{C}$ is the usual set of all complex numbers. By $\mathbf{A} \in \mathbb{R}^{m\times n}$ we mean $\mathbf{A}$ is an $(m \times n)$ real-valued matrix, and similarly for $\mathbf{A} \in \mathbb{C}^{m\times n}$. $\mathrm{tr}(\mathbf{A})$ and $\det(\mathbf{A})$ are respectively the trace and determinant of square matrix $\mathbf{A}$. $\mathrm{row}(\mathbf{A})$ and $\mathrm{col}(\mathbf{A})$ are the row and column spaces of $\mathbf{A}$ over a proper field, respectively. For any map T over vector spaces, we use $\ker(T)$, $\mathrm{rank}(T)$ and $\mathrm{nullity}(T)$ for the kernel, rank and nullity of T, respectively. $f \circ g = f(g)$ denotes the composition of functions f and g. $u(x)$ is the unit-step function defined as $u(x) = 1$ if $x \geq 0$ and $u(x) = 0$ if $x < 0$.

一、 Let

$$V = \left\{ [x_1, x_2, x_3, x_4]^\top \in \mathbb{R}^4 \ : \ x_1 - 2x_2 + x_3 = 0 \right\}$$

be a subspace of the real vector space $\mathbb{R}^4$, and let $\mathbf{A} \in \mathbb{R}^{4\times 4}$ be a real-valued matrix whose column space equals V. Which of the following statements is/are true?

(A) The dimension of $\mathrm{col}(\mathbf{A})$ is 3.

(B) The dimension of the orthogonal complement of $\mathrm{col}(\mathbf{A})$ is 1.

(C) For every vector $\underline{b} \in \mathbb{R}^4$, there exists $\underline{x} \in \mathbb{R}^4$ such that $\mathbf{A}\underline{x} = \underline{b}$.

(D) The orthogonal projection of vector $\underline{b} = [6, 0, 0, 0]^\top$ on $\mathrm{col}(\mathbf{A})$ is $[5, 2, -1]^\top$.

(E) None of the above.

參考用

注意:背面有試題

二、 Let $\mathbf{A} \in \mathbb{R}^{3\times 3}$ be a matrix such that

$$\text{col}(\mathbf{A}) \supseteq \text{span}\left\{\underline{v}_1 = [1,2,3]^\top,\ \underline{v}_2 = [-2,1,0]^\top,\ \underline{v}_3 = [1,0,1]^\top\right\}.$$

Which of the following statements is/are true?

(A) The vectors $\underline{v}_1$, $\underline{v}_2$ and $\underline{v}_3$ are linearly independent over field $\mathbb{R}$.

(B) $\det(\mathbf{A}) = 0$.

(C) The matrix $\mathbf{A}^\top$ has a multiplicative inverse.

(D) The vectors $\underline{v}_1$, $\underline{v}_2$ and $\underline{v}_3$ form a basis for $\mathbb{R}^3$.

(E) None of the above.

三、 Consider the following matrix multiplication

$$\mathbf{A} = \mathbf{LU}, \quad \text{where } \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{U} = \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Which of the following statements is/are true?

(A) The reduced row echelon form of $\mathbf{A}$ has two pivots, one at the first column and the other at the third column. Thus, $\text{rank}(\mathbf{A}) = 2$.

(B) As $\mathbf{L}^{-1}\mathbf{A} = \mathbf{U}$, the bottom row of $\mathbf{L}^{-1}$ performs a linear operation on the rows of $\mathbf{A}$ to yield the bottom, all-zero, row of $\mathbf{U}$. Therefore, the bottom row of $\mathbf{L}^{-1}$ can be a basis element for the left null space of $\mathbf{A}$.

(C) For the general case of $\mathbf{B} = \mathbf{E}^{-1}\mathbf{R}$ for some matrices $\mathbf{B}, \mathbf{R} \in \mathbb{R}^{3\times 4}$ and some invertible matrix $\mathbf{E} \in \mathbb{R}^{3\times 3}$, if there are two all-zero rows in $\mathbf{R}$, then the corresponding two rows of $\mathbf{E}$ can be basis elements for the left null space of $\mathbf{B}$.

(D) $\dim(\text{col}(\mathbf{A})) + \dim(\text{row}(\mathbf{A})) = 3$.

(E) None of the above.

四、Let $\mathbf{P}$ be a permutation matrix given as below that operates on the rows of a symmetric matrix $\mathbf{A}$

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}, \quad \mathbf{PA} = \begin{bmatrix} 4 & 2 & 6 \\ 5 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}.$$

It is observed that the product matrix $\mathbf{PA}$ is no longer symmetric; however, let $\mathbf{Q}$ be another permutation matrix such that the product matrix $\mathbf{QPA}$ is symmetric. Which of the following statements is/are true?

(A) $\mathbf{Q} = \mathbf{P}$

(B) $\mathbf{Q} = \mathbf{P}^{-1} = \mathbf{P}^{\top}$

(C) If $\mathbf{D} \in \mathbb{R}^{3\times 3}$ is a diagonal matrix, then so is $\mathbf{PDP}^{-1}$.

(D) $\mathbf{PAQ} = \mathbf{QAP}$

(E) None of the above.

五、Let $\mathbf{C}$ be the cofactor matrix of the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 9 \\ 2 & 2 & 8 \end{bmatrix}.$$

Which of the following statements is/are true?

(A) Every column vector of $\mathbf{C}^{\top}$ is in the right null space of $\mathbf{A}$.

(B) The dimension of the null space of $\mathbf{A}$ is 2.

(C) If $\text{rank}(\mathbf{A}) = r$, then there exists an $(r \times r)$ submatrix of $\mathbf{A}$ that is invertible.

(D) $\mathbf{C}$ is invertible.

(E) None of the above.

注意:背面有試題

※請在答案卡內作答

六、 Given the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & 1 \\ 1 & -1 & 1 & 3 \end{bmatrix}$$

which of the following statements is/are true?

(A) One of the eigenvalues of $\mathbf{A}$ is zero.

(B) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation given by $T(\underline{x}) = \mathbf{A}\underline{x}$; then T satisfies $T \circ T \circ T - 8T \circ T = -16T$.

(C) For any nonsingular matrix $\mathbf{S} \in \mathbb{R}^{4\times4}$, we have $\det(2\mathbf{I}_4 - \mathbf{SAS}^{-1}) = 16$

(D) $\mathbf{A}$ is not non-negative definite.

(E) None of the above.

七、 Continued from Problem 六, which of the following statements is/are true?

(A) The real vector space $\mathbb{R}^4$ together with the bilinear function $Q : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ given by $Q(\underline{x}, \underline{y}) = \underline{y}^\top \mathbf{A}\underline{x}$ is an inner product space.

(B) Let V be the vector space (over field $\mathbb{C}$) consisting of all (4×4) complex-valued circulant matrices that are orthogonal to $\mathbf{A}$ with respect to the inner product $\langle \mathbf{C}, \mathbf{D} \rangle = \mathrm{tr}(\mathbf{D}^\dagger \mathbf{C})$ for $\mathbf{C}, \mathbf{D} \in \mathbb{C}^{4\times4}$. Then the vector space V has dimension 3 over field $\mathbb{C}$.

(C) Let $\hat{\underline{b}}$ be the orthogonal projection of the vector $\underline{b} = [1, 0, 0, -1]^{\top}$ on the column space of **A**. Then $\|\hat{\underline{b}}\| = 1$.

(D) Continued from part (C), $\mathbf{A}\underline{b} = \mathbf{A}\hat{\underline{b}}$.

(E) None of the above.

八、 Let $V = \mathbb{R}^{2\times 3}$ be a vector space over field $\mathbb{R}$, and let $T : V \to V$ be a map given by

$$T(\mathbf{M}) = \mathbf{AMB}, \quad \text{where } \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Which of the following statements is/are true?

(A) T is an invertible linear operator on V.

(B) One of the eigenvalues of T is 9.

(C) Let $[T]_{\mathcal{B}}$ be the matrix for T relative to some basis $\mathcal{B}$ for V. Then $\mathrm{tr}([T]_{\mathcal{B}}) \neq 0$.

(D) Let $f(x)$ be a nonzero polynomial such that $\ker(f(T)) = V$. Then $f(x)$ must have degree at least four.

(E) None of the above.

※請在答案卡內作答

九、Consider the linear operator T defined on the vector space $V = \mathbb{C}^{n\times n}$ by $T(\mathbf{B}) = \mathbf{AB}-\mathbf{BA}$ for some complex-valued matrix $\mathbf{A} \in V$. Assuming all the eigenvalues of $\mathbf{A}$ are distinct (and $\mathbf{A}$ could be singular), which of the following statements is/are true?

(A) $\text{rank}(T) \leq n^2 - n$.

(B) $\text{nullity}(T) > n$.

(C) For every $\mathbf{B} \in \ker(T)$, $\mathbf{A}$ and $\mathbf{B}$ are simultaneously diagonalizable.

(D) The linear operator T is diagonalizable.

(E) None of the above.

十、Let $V = \mathbb{R}^2$ be an inner product space in which the inner product is defined as

$$H_{\mathbf{A}}(\underline{x}, \underline{y}) = \underline{y}^{\top}\mathbf{A}\underline{x}, \quad \text{where } \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

With respect to this inner product $H_{\mathbf{A}}$, let the QR-decomposition of the identity matrix $\mathbf{I}_2$ be $\mathbf{I}_2 = \mathbf{QR}$ such that the columns of $\mathbf{Q}$ are orthonormal to each other, and $\mathbf{R}$ is an upper triangular matrix with positive diagonal entries. Which of the following statements is/are true?

(A) The rows of $\mathbf{Q}$ form an orthonormal basis for V.

(B) $\det(\mathbf{R}) = \sqrt{3}$.

(C) $\text{tr}(\mathbf{R}^{\top}\mathbf{R}) = 3$.

注意:背面有試題

(D) The second column vector of $\mathbf{R}$ has a smaller norm (with respect to $H_{\mathbf{A}}$) than the first column vector of $\mathbf{R}$.

(E) None of the above.

十一、 Consider the following first-order differential equation for $y(x)$

$$y'(x) = \frac{y(x) - x}{y(x) + x}. \tag{1}$$

Which of the following statements is/are true?

(A) Equation (1) is a linear differential equation.

(B) Equation (1) is an exact differential equation.

(C) It is possible for having a solution with an initial value $y(-1) = 0$.

(D) It is possible for having a solution with an initial value $y(0) = 1$.

(E) None of the above.

十二、 Consider the following second-order differential equation

$$x^2y''(x) - x(x+2)y'(x) + (x+2)y(x) = f(x). \tag{2}$$

For the homogeneous solution, i.e. when $f(x) = 0$, if given one solution $y_1(x) = x$, a second linearly independent solution $y_2(x)$ can be derived by setting $y_2(x) = v(x)y_1(x)$. Which of the following statements is/are true?

(A) $x^3v''(x) - x^3v'(x) = 0$

(B) $v''(x) = v'(x)$

(C) $v'(x) = e^x$

(D) $v(x) = e^x$

(E) None of the above.

十三、 Continued from Problem 十二, given $f(x) = x^3$ in equation (2), use the method of variation of parameters to find "the particular solution" $y_p(x)$ for $y(x)$, i.e., set $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, with $y_1(x)$ and $y_2(x)$ obtained from Problem 十二. Which of the following statements is/are true?

(A) $u_1'(x) = -1$

(B) $u_1'(x) = -x^2$

(C) $u_2'(x) = e^{-x}$

(D) $u_2'(x) = x^2 e^{-x}$

(E) None of the above.

十四、 Let $\underline{x}(t) = [x_1(t), x_2(t)]^\top$ and consider the following second-order system

$$\underline{x}''(t) = \mathbf{A}\,\underline{x}(t), \qquad \text{where } \mathbf{A} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}. \tag{3}$$

Equation (3) can be rewritten as a first order system by setting $\underline{y}(t) = [x_1(t), x_1'(t), x_2(t), x_2'(t)]^\top$ to yield

$$\underline{y}'(t) = \mathbf{B}\,\underline{y}(t), \qquad \text{where } \mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$$

for some $\mathbf{B}_{11}, \mathbf{B}_{12}, \mathbf{B}_{21}, \mathbf{B}_{22} \in \mathbb{R}^{2\times 2}$. Which of the following statements is/are true?

(A) $\mathbf{B}_{11} = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}$

(B) $\mathbf{B}_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

注意:背面有試題

(C) $\mathbf{B}_{21} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

(D) $\mathbf{B}_{22} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$

(E) None of the above.

十五、 Continued from Problem 十四, which of the following statements is/are true regarding the exponential matrix $e^{\mathbf{A}}$?

(A) $e^{\mathbf{A}} = \begin{bmatrix} e^{-3} & e \\ e^{2} & e^{-2} \end{bmatrix}$

(B) $e^{\mathbf{A}} = \begin{bmatrix} e^{-1} & e^{-4} \\ 2e^{-1} & -e^{-4} \end{bmatrix}$

(C) $e^{\mathbf{A}} = \frac{1}{3}\begin{bmatrix} e^{-1} + 2e^{-4} & e^{-1} - e^{-4} \\ 2e^{-1} - 2e^{-4} & 2e^{-1} + e^{-4} \end{bmatrix}$

(D) $e^{\mathbf{A}} = \frac{1}{3}\begin{bmatrix} e^{-1} - 2e^{-4} & e^{-1} + e^{-4} \\ 2e^{-1} + 2e^{-4} & 2e^{-1} - e^{-4} \end{bmatrix}$

(E) None of the above.

※請在答案卡內作答

十六、 Consider the following differential equation:

$$3xy''(x) + (2-x)y'(x) - y(x) = 0$$

with $y(0) = 1$ and $y'(0) = \frac{1}{2}$. Which of the following statements is/are true?

(A) $x = 0$ is an ordinary point.

(B) The radius of convergence for the series solution is 1.

(C) $y''(x) = \frac{1}{5}$.

(D) $|y(1)| > \frac{3}{2}$.

(E) None of the above.

注意:背面有試題

十七、Consider the following differential equation:

$$tg''(t) + g'(t) + 4tg(t) = 0$$

with $g(0) = 1$ and $g'(0) = 0$. Let $G(s)$ denote the unilateral Laplace transform of $g(t)$. Which of the following statements is/are true?

(A) $\lim_{t \to 0} g''(t) = -2$.

(B) $g(1) = \frac{1}{4}$.

(C) $G(\sqrt{5}) = \frac{1}{3}$.

(D) $\lim_{s \to \infty} sG(s) = 1$.

(E) None of the above.

※請在答案卡內作答

十八、 Consider the differential equation

$$y'(t) + y(t) = \frac{5}{2}\sin(2t)u(100t)$$

with $y(0) = 0$. Which of the following statements is/are true?

(A) $y(\pi) = -1$.

(B) $y(\frac{\pi}{2}) = 1$.

(C) $y'(\pi) = 1$.

(D) $y'(\frac{\pi}{2}) = \frac{1}{\sqrt{2}}$.

(E) None of the above.

十九、 Consider $f(x) = x$, for $0 < x < 1$. Let $A(x)$ and $B(x)$ denote the half-range cosine and sine series expansions of $f(x)$, respectively. Which of the following statements is/are true?

(A) $A(x) = \sum_{n=0}^{\infty} a_n \cos(\frac{n\pi x}{2})$ for some a_n.

(B) $B(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n\pi}\sin(n\pi x)$.

(C) $B(x) = \sum_{n=1}^{\infty}\frac{(-1)^n}{2n\pi}\sin(n\pi x)$.

(D) $A(99.5) + B(-9.5) = 1$.

(E) None of the above.

注意:背面有試題

二十、 Consider the following boundary-value problem for the bivariate function $v(x,t)$:

$$\frac{\partial^2 v(x,t)}{\partial x^2} + x^2 = \frac{\partial^2 v(x,t)}{\partial t^2},\ 0 < x < 1,\ t > 0$$
$$v(0,t) = 1,\ v(1,t) = 0,\ t > 0$$
$$v(x,0) = -\frac{1}{12}x^4 + \frac{1}{12}x + 1,\ \left.\frac{\partial v(x,t)}{\partial t}\right|_{t=0} = 0,\ 0 < x < 1.$$

Which of the following statements is/are true?

(A) $\left.\frac{\partial v(x,t)}{\partial t}\right|_{x=\frac{1}{2},t=1} = 1.$

(B) $\left.\frac{\partial v(x,t)}{\partial t}\right|_{x=\frac{1}{3},t=2} = 0.$

(C) $\left.\frac{\partial^2 v(x,t)}{\partial x \partial t}\right|_{x=\frac{1}{2},t=1} = 0.$

(D) $\left.\frac{\partial^2 v(x,t)}{\partial x \partial t}\right|_{x=\frac{1}{3},t=1} = 1.$

(E) None of the above.