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Note: Detailed derivations are required to obtain a full score for Problem 2 to Problem 4.

1. (15%) Among the 10 statements below, only 5 are true and the other 5 are false. Find out which 5 are true. (You are not obligated to give explanations, but **will get zero point if listing more than 5 of them**).

(**a**) Let V be a vector space and $S \subseteq \mathsf{V}$ be a subset. Then, $\text{span}(S)$ is the intersection of all subspaces of V that contain S.

(**b**) Let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be a linear transformation. Let $S = \{v_1, v_2, ..., v_n\}$ be a subset of V. If S is linearly dependent, its image $\mathsf{T}(S)$ is also linearly dependent.

(**c**) The basis of any vector space uniquely exists.

(**d**) Let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be a linear transformation. If T is invertible, then $\dim(\mathsf{V}) = \dim(\mathsf{W})$.

(**e**) Let $A \in \mathsf{M}_{m\times n}(\mathbb{R})$ be an arbitrary matrix. If $m < n$, then $\text{rank}(A) > \text{rank}(A^t)$.

(**f**) Assume that $A \in \mathsf{M}_{m\times n}(\mathbb{R})$ and $b \in \mathsf{M}_{m\times 1}(\mathbb{R})$. Let x_1 and x_2 be two column vectors in $\mathbb{R}^n$. If $x_1 \neq x_2$ and $Ax_1 = b = Ax_2$, then the system of linear equations $Ax = b$ has infinitely many solutions.

(**g**) Let A and B be square matrices of the same size. If $AB = O$, then $\mathsf{R}(\mathsf{L}_B) \supseteq \mathsf{N}(\mathsf{L}_A)$. (*Remarks*: L_A and L_B denote the linear transformation of matrix multiplication from the left.)

(**h**) Let A and B be square matrices of the same size. If $AB = A$, then $B = I$.

(**i**) Let A be a square matrix and $r \in \mathbb{R}$. Then, $\det(rA) = r\det(A)$.

(**j**) Assume that $A \in \mathsf{M}_{3\times 3}(\mathbb{C})$ and $A^tA = -I$. Then, the entries in A cannot all be real numbers.

2. (10%) Define a linear transformation $\mathsf{T}: \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathsf{T}\big((1,0,0)\big) = (0,1,0)$, $\mathsf{T}\big((0,1,0)\big) = (0,0,1)$, and $\mathsf{T}\big((0,0,1)\big) = (1,0,0)$.

(**a**) (5%) Find a vector $u = (u_x, u_y, u_z)$ such that $\mathsf{T}(u) = u$ and $\sqrt{u_x^2 + u_y^2 + u_z^2} = 1$.

(**b**) (5%) Is $\mathsf{T}: \mathbb{R}^3 \to \mathbb{R}^3$ one-to-one and onto? Why or why not?

3. (15%) Let $\mathsf{W}_1, \ldots, \mathsf{W}_k$ be subspaces of a vector space V. The **direct sum** V of $\mathsf{W}_1, \ldots, \mathsf{W}_k$ is defined if the following two conditions hold.

$$\mathsf{V} = \sum_{i=1}^{n} \mathsf{W}_i \qquad \text{and} \qquad \mathsf{W}_j \cap \sum_{i\neq j} \mathsf{W}_i = \{0\}\ \forall j\ (1 \leq j \leq k)$$

If the two conditions hold, then V is denoted by $\mathsf{V} = \mathsf{W}_1 \oplus \ldots \oplus \mathsf{W}_k$. Prove or disprove (by providing a counterexample) of the following statements.

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(a) (7%) If $V = W_1 \oplus \ldots \oplus W_k$. Then, for any distinct i and j, W_i and W_j intersect at exactly the zero vector.

(b) (8%) If $V = \sum_{i=1}^{n} W_i$, and W_i and W_j intersect at exactly the zero vector for any distinct i, j $(1 \le i, j \le k)$. Then V is the direct sum of $W_1, \ldots, W_k$

4. (10%) Let V be a finite-dimensional complex inner product space and $T : V \to V$ be a linear operator. T is normal if and only if $TT^* = T^*T$, where T^* is the adjoint of T. Moreover, T is **nilpotent** if there exists $n \in \mathbb{N}$ such that T^n is the zero operator. Prove the following statement. If T is both normal and nilpotent, then T is the zero operator itself.

5 (5%)

Solve $y'' + 5y' + 4y = e^{-x}$

6. (5%)

Solve the following differential equation with the initial conditions by using Laplace transform.

$y'' + y = \delta(t - \pi)$

$y(0) = 0$

$y'(0) = 0$

7. (5%) $y'' + (1 + \lambda)y = 0$; $y(0) = y(\pi) = 0$

Find the eigenvalues and eigenfunctions.

8. (5%)

$$f(t) = \begin{cases} e^{-at}, & t \ge 0 \\ 0, & t < 0 \end{cases} \quad (\alpha > 0)$$

Find the Fourier Transform of $f(t)$.

9. (5%)

$$f(t) = \begin{cases} -1, & -\pi < t < 0 \\ 1, & 0 < t < \pi \end{cases}$$

$f(t + n2\pi) = f(t)$

Find the Fourier Series of $f(t)$.

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10. (5%)

Find u and v such that $f(z) = u(x,y) + iv(x,y)$;

determine where f is differentiable and where f is not.

$$f(z) = \left|z^2\right|$$

11. (5%)

Find the principle value of $(i)^{1-2i}$

12. (5%)

Find the residue of $f(z) = \dfrac{\sin z}{z^4(z^2+i)}$ at z = 0.

13. (5%)

Consider the complex series $f(z) = \sum_{n=0}^{\infty} \dfrac{z^n}{4^{n+1}} + \sum_{n=1}^{\infty} \dfrac{1}{z^n}$.

Find the region of convergence.

14. (5%)

$$\oint_C e^{3/z} dz$$

where C: $|z-i| = 2$, clockwise