類組：電機類　科目：工程數學C(3005)　※請在答案卡內作答

- 本測驗試題為多選題，請選出所有正確的答案，並請用2B鉛筆作答於答案卡。
- 共二十題，每題五分，每答對一個選項，可得一分，每答錯一個選項，倒扣一分，直到本科分數扣完為止。未作答則該題不給分也不扣分。

Notation: In the following questions, boldface lowercase letters such as $\mathbf{x}$, $\mathbf{v}$, etc. denote column vectors of proper length; boldface uppercase letters such as $\mathbf{A}$, $\mathbf{B}$, etc. denote matrices of proper size; $\mathbf{A}^T$ means the transpose of matrix $\mathbf{A}$. C(A) is the column space of matrix A, and N(A) is the null space of matrix A. $\mathbb{R}$ is the usual set of all real numbers.

一、Forward elimination uses possible elementary row operations to do elimination on a matrix. Suppose a block lower triangular matrix $\mathbf{E}=\begin{bmatrix}\mathbf{H} & \mathbf{I}\\ \mathbf{J} & \mathbf{K}\end{bmatrix}$ can do elimination on a whole block column of matrix $\mathbf{A}=\begin{bmatrix}\mathbf{M} & \mathbf{N}\\ \mathbf{P} & \mathbf{Q}\end{bmatrix}$ to produce an upper triangular matrix $\mathbf{U}=\begin{bmatrix}\mathbf{Q} & \mathbf{R}\\ \mathbf{S} & \mathbf{T}\end{bmatrix}$. If $\mathbf{H}$ and $\mathbf{K}$ are identity matrices, and $\mathbf{A}$ has independent columns, which of the following statements are true?

(A) $\mathbf{A}^T\mathbf{A}$ is invertible.
(B) $\mathbf{T}=\mathbf{Q}-\mathbf{P}\mathbf{M}^{-1}\mathbf{N}$.
(C) $\mathbf{J}=\mathbf{P}\mathbf{M}^{-1}$.
(D) $\mathbf{Q}=\mathbf{M}$; $\mathbf{R}=-\mathbf{N}$.
(E) $\mathbf{E}$ is invertible.

二、Given three vectors, $\mathbf{v}_1=[1\ \ 2\ \ 3\ \ 4\ \ 5]^T$, $\mathbf{v}_2=[6\ \ 7\ \ 8\ \ 9\ \ 10]^T$, and $\mathbf{v}_3=[2\ \ -3\ \ 0\ \ 1\ \ 0]^T$, which of the following statements are true ?

(A) The set of all linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ is isomorphic to $\mathbb{R}^3$.

(B) If S is a subspace spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, $\mathbf{v}_3$ can be a vector in one of the bases for $S^{\perp}$, which is perpendicular to S.

(C) If $\mathbf{A}=[\mathbf{v}_1\ \ \mathbf{v}_2]$, the rank of $\mathbf{A}$ is 3.

(D) If $\mathbf{A}=[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$, the dimension of N(A) is 2.

(E) If $\mathbf{A}=[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$, the column vectors of $\mathbf{A}^T$ are linearly independent.

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三、S_1 is the subspace generated by $\mathbf{v}_1 = [1 \quad -1 \quad 1 \quad 1]^T$, and S_2 is the subspace generated by $\mathbf{v}_2 = [1 \quad 0 \quad -1 \quad 0]^T$. If $\mathbf{A} = [\mathbf{v}_1 \quad \mathbf{v}_2]$, which of the following is within the subspace generated by $\mathbf{v}_1$ and $\mathbf{v}_2$?

(A) $S_1 \cap S_2$.

(B) $S_1 \cup S_2$.

(C) C(**A**).

(D) $[4 \quad -4 \quad 3 \quad 4]^T + c_1 * [1 \quad -1 \quad 1 \quad 1]^T + c_2 * [1 \quad 0 \quad -1 \quad 0]^T$, where c_1 and c_2 are arbitrary real numbers.

(E) $[x_1 \quad x_2 \quad x_3 \quad x_4]^T$, where $\sum_{i=1}^{4} x_i = 0$.

四、Given a subspace W: $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} + d\mathbf{u} = 0$, $\{a,b,c,d \in \mathbb{R}^1\}$; If the orthogonal projection matrix onto the W is **P**, which of the following statements are always true?

(A) $\mathbf{P}^T = \mathbf{P}$.

(B) $\mathbf{P} = \mathbf{P}^{-1}$.

(C) The column vectors of $\mathbf{P}^T$ are orthogonal to each other.

(D) $\mathbf{P}^2 = \mathbf{P}$.

(E) $\mathbf{P}^T\mathbf{P}$ is invertible.

五、**A** is an m by n matrix and $\mathbf{Q} = [\mathbf{q}_1 \quad \cdots \quad \mathbf{q}_n]$, suppose **Q** has orthonormal columns, i.e. $\mathbf{q}_i$ (i=1~n) are orthonormal vectors, and **A**=**QR**, which of the following statements are true?

(A) $\mathbf{Q}^T = \mathbf{Q}^{-1}$.

(B) The orthogonal projection matrix onto C(**Q**) is Identity Matrix.

(C) $\mathbf{q}_i^T \mathbf{q}_j = 0$ if $i \neq j$.

(D) $\det(\mathbf{Q}^T\mathbf{Q}) = 1$.

(E) If m=n, det(**A**)= det(**R**).

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六、For matrix $\mathbf{A}=\begin{bmatrix}1 & -1\\ -2 & 2\\ 2 & -2\end{bmatrix}$, which of the following statements are true for each basis and dimension of four fundamental subspaces? Note the coefficients $a,\ b,\ c \in \mathbb{R}^1$.

(A) Column space of matrix **A**: $\mathrm{C}(\mathbf{A}) = a\dfrac{\begin{bmatrix}1\\-2\\2\end{bmatrix}}{3}$, dimension $\mathrm{C}(\mathbf{A}) = 1$.

(B) Null space of matrix **A**: $\mathrm{N}(\mathbf{A}) = b\dfrac{\begin{bmatrix}1\\-2\\2\end{bmatrix}}{3}$, dimension $\mathrm{N}(\mathbf{A}) = 1$.

(C) Column space of matrix $\mathbf{A}^{\mathrm{T}}$: $\mathrm{C}(\mathbf{A}^{\mathrm{T}}) = c\dfrac{\begin{bmatrix}1\\-1\end{bmatrix}}{\sqrt{2}}$, dimension $\mathrm{C}(\mathbf{A}^{\mathrm{T}}) = 1$.

(D) Matrix **A** has 2 eigenvalues and eigenvectors.

(E) Rank of matrix **A** is 2.

七、Which of the following statements are true?

(A) A real matrix with real eigenvalues and eigenvectors is symmetric.

(B) A real matrix with real eigenvalues and complete set of orthogonal eigenvectors is symmetric.

(C) The inverse of a symmetric matrix is symmetric.

(D) If the columns of a square matrix **S** are linearly independent, **S** is invertible.

(E) A matrix with complete set of independent eigenvectors is diagonalizable.

八、Find the eigenvalues and eigenvectors of $\mathbf{A}^3$, where

$$\mathbf{A}=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}.$$

(A) $\lambda_1 = 1, \lambda_2 = 3;\ \mathbf{x}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$.

(B) $\lambda_1 = -1, \lambda_2 = 3;\ \mathbf{x}_1 = \begin{bmatrix}1\\1\end{bmatrix},\ \mathbf{x}_2 = \begin{bmatrix}-1\\1\end{bmatrix}$.

(C) $\lambda_1 = 1, \lambda_2 = 9;\ \mathbf{x}_1 = \begin{bmatrix}1\\1\end{bmatrix},\ \mathbf{x}_2 = \begin{bmatrix}-1\\1\end{bmatrix}$.

(D) $\lambda_1 = 1, \lambda_2 = 27;\ \mathbf{x}_1 = \begin{bmatrix}1\\1\end{bmatrix},\ \mathbf{x}_2 = \begin{bmatrix}-1\\1\end{bmatrix}$.

(E) $\lambda_1 = 1, \lambda_2 = 1;\ \mathbf{x}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}-1\\1\end{bmatrix}$.

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九、The rabbit and wolf population shows fast growth of rabbits (from $6r$) but loss to wolves (from $-2w$)

$$r' = 6r - 2w$$
$$w' = 2r + w$$

Which of the following statements are true?

(A) The eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 2$.

(B) If $r(0) = w(0) = 30$, the population of rabbit at time t becomes: $20e^{5t} + 10e^{2t}$.

(C) If $r(0) = w(0) = 30$, the population of wolf at time t becomes: $10e^{-2t} + 20e^{-5t}$.

(D) After a long time, the ratio of wolves to rabbits approaches 2.

(E) After a long time, the wolves will be extinct due to lack of food.

十、For a real matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$, which of the following statements are true?

(A) $\mathbf{A}$ is positive definite if $b = 3$.

(B) The eigenvalues in positive definite matrix must be positive.

(C) $\mathbf{A}$ equals $\mathbf{R}^T\mathbf{R}$ for a matrix $\mathbf{R}$ with independent columns if $b = 0$.

(D) All 3 pivots in matrix $\mathbf{A}$ are positive if $b = 0$.

(E) The determinant of $\mathbf{A}$ is always positive for all b.

十一、Suppose that $y_1(x),\cdots,y_n(x)$ are n-1 times differentiable functions over $(-\infty,\infty)$ and $W(x)$ denotes the Wronskian of $y_1(x),\cdots,y_n(x)$ at x. Which of the following statements are true?

(A) $W(x)$ vanishes at every x if $y_1(x),\cdots,y_n(x)$ are linearly dependent.

(B) If $y_1(x),\cdots,y_n(x)$ are linearly independent, then there is x such that $W(x) \neq 0$.

(C) If $y_1(x),\cdots,y_n(x)$ are linearly independent, then $W(x) \neq 0$ for all x.

(D) If $y_1(x),\cdots,y_n(x)$ are also solutions of an nth-order linear homogeneous ordinary differential equation with constant coefficients. Then either $W(x) \neq 0$ for all x or $W(x) = 0$ for all x.

(E) None of the above statements are true.

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十二、 Suppose that $p(x)$, $q(x)$, and $f(x)$ are continuous functions over $(-\infty,\infty)$, and $f(x)$ is not the zero function. Please determine which of the solution sets of the following differential equations are vector spaces?

(A) $y''(x)+p(x)y'(x)+q(x)y(x)=f(x)$.

(B) $y''(x)+p(x)y'(x)+q(x)y(x)=0$.

(C) $y(x)y''(x)+y(x)=0$.

(D) $x^2y''(x)+5xy'(x)+12y(x)=1$.

(E) $(y'(x))^3=1$.

十三、 Consider the differential equation $y''(x)+ay'(x)+by(x)=0$. Which of the following statements are true?

(A) $y(x)\to 0$ no matter what $y(0)$ and $y'(0)$ if $a>0$ and $b>0$.

(B) $y(x)$ is bounded no matter what $y(0)$ and $y'(0)$ if $a>0$ and $b=0$.

(C) $y(x)$ is unbounded for all $(y(0),y'(0))\neq(0,0)$ if $\min\{a,b\}<0$.

(D) $y(x)$ is always unbounded whenever $(y(0),y'(0))\neq(0,0)$ if $ab<0$.

(E) None of the above statements are true.

十四、 Which of the following statements are true?

(A) $x(t)=\int_0^t \tau e^{-\tau} f(t-\tau)d\tau$ is the solution of $x''(t)+2x'(t)+x(t)=f(t)$ and $x(0)=x'(0)=0$.

(B) Let $\delta(x)$ be the impulse function. Then the following two initial value problems
$y''(x)+ay'(x)+by(x)=f(x),\ y(0)=0\ \ \&\ y'(0)=v$ and
$y''(x)+ay'(x)+by(x)=f(x)+v\delta(x),\ y(0)=0\ \ \&\ y'(0)=0$ have the same solution for $x>0$.

(C) If $f(x)$ is a continuous function over $[0,\infty)$, then there always exists some complex number s such that the Laplace transform $\int_0^\infty e^{-sx}f(x)dx$ converges.

(D) Inverse Laplace transformation, provided it exists, is not a linear function.

(E) None of the above statements are true.

十五、 Consider the initial value problem: $y'(x)=x^2y(x)(1-y(x))^3,\ y(0)=0.5$. Which of the following statements are true?

(A) The initial value problem may have infinite many solutions.

(B) The solution, if exists, always lies between 0 and 1.

(C) The given differentiable equation is separable.

(D) The given differentiable equation is exact.

(E) None of the above statements are true.

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十六、In the following choices of (A) to (E), please find the solutions to the given system of equations

$$\frac{d}{dt}\mathbf{x}=\begin{bmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{bmatrix}\mathbf{x}+\begin{bmatrix}e^{t}\\ \sqrt{3}e^{-t}\end{bmatrix}.$$

(A) $5\begin{bmatrix}\sqrt{3}\\ 1\end{bmatrix}e^{-2t}+\begin{bmatrix}2/3\\ 1/\sqrt{3}\end{bmatrix}e^{t}-\begin{bmatrix}-1\\ 2/\sqrt{3}\end{bmatrix}e^{-t}.$

(B) $3\begin{bmatrix}1\\ -\sqrt{3}\end{bmatrix}e^{-2t}-\begin{bmatrix}2/3\\ 1/\sqrt{3}\end{bmatrix}e^{t}+\begin{bmatrix}-1\\ 2/\sqrt{3}\end{bmatrix}e^{-t}.$

(C) $5\begin{bmatrix}-\sqrt{3}\\ 1\end{bmatrix}+3\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}e^{-2t}+\begin{bmatrix}2/3\\ -1/\sqrt{3}\end{bmatrix}e^{t}-\begin{bmatrix}1\\ 2/\sqrt{3}\end{bmatrix}e^{-t}.$

(D) $5\begin{bmatrix}-\sqrt{3}\\ 1\end{bmatrix}e^{2t}+3\begin{bmatrix}1\\ -\sqrt{3}\end{bmatrix}e^{-2t}-\begin{bmatrix}2/3\\ 1/\sqrt{3}\end{bmatrix}e^{t}+\begin{bmatrix}-1\\ 2/\sqrt{3}\end{bmatrix}e^{-t}.$

(E) $5\begin{bmatrix}\sqrt{3}\\ 1\end{bmatrix}e^{2t}+3\begin{bmatrix}1\\ -\sqrt{3}\end{bmatrix}e^{-2t}-\begin{bmatrix}2/3\\ 1/\sqrt{3}\end{bmatrix}e^{t}+\begin{bmatrix}-1\\ 2/\sqrt{3}\end{bmatrix}e^{-t}.$

十七、We can derive the solution as a series of normalized eigenfunctions of the corresponding homogeneous problem as follows:

$y''+\lambda y=0,\ y(0)=0,\ y(1)+y'(1)=0$

where the solution can be expressed as

$$y=\sum_{n=1}^{\infty}\frac{Cf(x)\sin\sqrt{\lambda_n}}{\lambda_n(\lambda_n-2)(1+\cos^2\sqrt{\lambda_n})}.$$

Please find the corresponding term of $Cf(x)$ from the following choices:

(A) $4(\sin\sqrt{\lambda_n}x+\cos\sqrt{\lambda_n}x)$

(B) $4\sin\sqrt{\lambda_n}x+\cos\sqrt{\lambda_n}x$

(C) $\sin\sqrt{\lambda_n}x+4\cos\sqrt{\lambda_n}x$

(D) $4\sin\sqrt{\lambda_n}x$

(E) $2\cos\sqrt{\lambda_n}x$

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十八、 Continuing on the previous question, please find the value of the λ_n for $n \geq 4$.

(A) $\frac{(2n+1)^2 \pi^2}{4}$

(B) $\frac{(n\pi)^2}{2}$

(C) $\frac{(n\pi+1)^2}{2}$

(D) $\frac{(2n-1)^2 \pi^2}{4}$

(E) $\frac{(n\pi-1)^2}{4}$

十九、 Find the inverse Laplace transform of the given functions:

$$L^{-1}\left\{\frac{2s+1}{4s^2+4s+5}\right\}$$

(A) $\frac{1}{4}e^{t/2}\sinh(t-2)$

(B) $\frac{1}{2}e^{-t/2}\sin t$

(C) $\frac{1}{2}e^{-t/2}\cos t$

(D) $\frac{1}{4}e^{t/2}\cosh(t-2)$

(E) $\frac{1}{4}e^{-t/2}\sinh t$

二十、 Assume that there is a Fourier series converging to the function f defined by

$$f(x)=\begin{cases}-x, & -2 \leq x < 0, \\ x, & 0 \leq x < 2;\end{cases} \quad \text{and} \quad f(x+4)=f(x).$$

Please find the following choices, which are the terms in this Fourier series?

(A) $\frac{-8}{\pi^2}\frac{\cos(\frac{15}{2}\pi x)}{225}$

(B) $\frac{1}{\pi^2}\frac{\sin(4\pi x)}{8}$

(C) 1

(D) $\frac{8}{\pi^2}\frac{\cos(\frac{5}{2}\pi x)}{25}$

(E) $\frac{1}{\pi^2}\frac{\sin(8\pi x)}{32}$