

類組：電機類 科目：工程數學 A(3003)

※請在答案卷內作答

1. (15 pts). Write down your proof as detailed as possible.

Let V be a finite-dimensional vector space over a field F . The dual space V^* of V is defined as the vector space of linear functionals on V , i.e. $V^* = \{f | f: V \rightarrow F\}$. Let $T: V \rightarrow V$ be a linear operator. T 's transpose $T^t: V^* \rightarrow V^*$ is a linear mapping from V^* to V^* defined by $T^t(g) = gT$ for each $g \in V^*$. Let $V = P(\mathbb{R})$, the vector space of polynomials over real numbers. For each positive integer k , define $\varphi_k: V \rightarrow \mathbb{R}$ by $\varphi_k(f(x)) = f^{(k)}(0)$, the k -th derivative of $f(x)$ at $x = 0$. Let $\partial: V \rightarrow V$ be the differentiation mapping defined by $\partial(f(x)) = f'(x)$. Prove that $\partial^t \varphi_k = \varphi_{k+1}$.

2. (15 pts). Let $V = C^\infty$, the vector space of all real functions having derivatives of all orders, and let y_1, y_2, \dots, y_n be some fixed linearly independent functions in V . Let $\delta: M_{n \times n}(\mathbb{R}) \rightarrow \mathbb{R}$ be an alternating n -linear function (defined for each $n \times n$ matrix over \mathbb{R}) that is not identically to zero. For each $y \in V$ and $t \in \mathbb{R}$, define $T(y(t)) \in \mathbb{R}$ as follows.

$$T(y(t)) = \delta \begin{pmatrix} y(t) & y_1(t) & y_2(t) & \cdots & y_n(t) \\ y'(t) & y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y^{(n)}(t) & y_1^{(n)}(t) & y_2^{(n)}(t) & \cdots & y_n^{(n)}(t) \end{pmatrix}$$

- (a) Prove that $T: V \rightarrow V$ is a linear transformation.
- (b) Prove that the null space of T satisfies $N(T) \supset \text{Span}(\{y_1, y_2, \dots, y_n\})$.

3. (15 pts). Eigenvalues and eigenvectors.

- (a) Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$. Find its eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$.
- (b) Continuing from above, find a matrix $Q \in M_{2 \times 2}(\mathbb{C})$ such that $Q^{-1}AQ$ is diagonal.
- (c) Find the minimum positive integer n such that $A^n = I$.

4. (10 pts). Least-square approximation. Let $f(t)$ be defined as follows,

$$f(t) = \begin{cases} 1, & \text{if } 0 \leq t < \pi \\ -1, & \text{if } -\pi < t < 0. \end{cases}$$

Also, define

$$g(t) = a \cos t + b \cos 2t + c \sin t.$$

Find the coefficients (a, b, c) such that $E = \int_{-\pi}^{\pi} |g(t) - f(t)|^2 dt$ is minimized.

參考用

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5. (25 pts). For a system of non-linear ordinary equations (with $m > 0$) in a two dimension phase plane

$$\begin{aligned}y_1' &= y_2 - 2, \\y_2' &= \frac{2m}{\pi}y_1 - \sin y_1,\end{aligned}$$

- (a) For $m = 1$, please find all the critical points in the phase plane.
(b) Find the range for the value of m such that this system of ordinary differential equation has seven critical points.

6. (20 pts). Complex integrals

- (a) Find all the poles of

$$\frac{e^{az}}{1+e^z}, \quad \text{where } 0 < a < 1.$$

- (b) Calculate the integral for the contour $C : \text{Im}(z) > 0$, i.e., the Upper Half Plane (UHP), in the counter-clockwise direction,

$$\oint_{C:\text{UHP}} \frac{e^{az}}{1+e^z} dz, \quad \text{where } 0 < a < 1.$$

參考用

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