國　立　清　華　大　學　命　題　紙

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科目 機率論　科目代碼 2002、2202　共 3 頁第 1 頁　*請在【答案卷卡】內作答

I. Answer the following questions.

1. (20%) Suppose an urn has r red balls and b black balls. A ball is drawn and its color noted. Then the drawn ball together with c balls $(c>0)$ of the same color as the drawn ball are added to the urn. The procedure is repeated many times. Thus, at the kth draw, there are $b+r+c(k-1)$ balls in the urn and we assume that the probability of drawing any particular ball is $1/(b+r+c(k-1))$.

(a) (5%) Find the probability that the 2nd ball drawn is red.

(b) (5%) Given that the 2nd ball drawn is red, find the probability that the first ball is red.

(c) (5%) Given that the 2nd ball drawn is red, find the probability that the first ball is back.

(d) (5%) Find the probability that the 3rd ball drawn is red.

2. (15%) Let X and Y be independent random variables each having the uniform density on $\{0,1,\cdots,N\}$.

(a) (5%) Find $P(X>Y)$.

(b) (5%) Find the density of $\max(X,Y)$.

(c) (5%) Find the density of $X+Y$.

II. Answer the following questions.

1.(15%) Let X and Y be two random variables with their joint probability density function given by

$$f_{XY}(x,y)=\begin{cases} 6\times 10^{-6}\exp(-0.001x-0.002y) & 0<x<y \\ 0 & elsewhere \end{cases}$$

(a) Derive the marginal probability distribution of X. (b) Compute the mean of the random variable X. (c) Compute the probability that Y exceeds 2000 given that X takes the value 1500. Use e ≒ 2.7 in your calculation if needed.

2.(10%) X_1, X_2, …, X_n are independent continuous random variables with means μ_1, μ_2,…, μ_n and variances $\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2$, respectively. Let a random variable Y be defined by $Y=a_1X_1+a_2X_2+\cdots+a_nX_n$, where $a_1, a_2, \cdots, a_n$ are real numbers. (a) Compute the mean and variance of the random variable Y. (b) Assume the moment generating function for X_i is $\exp(\mu_i t+\frac{1}{2}\sigma_i^2 t^2)$ for i=1, …, n. Derive the moment generating function for Y.

3.(10%) Let X be a normally distributed random variable with zero mean and unit variance, i.e. its probability density function is given by

$$f_X(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

A random variable Y is defined to be $Y = X^2$. What is the probability density function for Y? Show your derivation.

III. Answer the following questions.

1.(10%)

Let X be a Rayleigh probability distribution having the following density function

$$f(x) = \frac{x}{s^2}\exp\left(\frac{-x^2}{2s^2}\right), \quad x \ge 0, \text{ where } s>0 \text{ is a fixed number,}$$

$$= 0, \text{ otherwise.}$$

(a) Compute the mean of X.

(b) Compute the variance of X.

Hint: Define $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$, then $\Gamma(\frac{1}{2})=\sqrt{\pi}$.

2.(20%)

A bucket contains four balls numbered from 1 through 4. The balls are selected one at a time without replacement. A match occurs if ball numbered k is the kth ball selected. Let the event A_i denote a match on the ith draw, where i = 1,2,3,4. Show that

(a) $P(A_i) = 3!/4!$, i=1,2,3,4.

(b) $P(A_i \cap A_j) = 2!/4!$, $1 \leqq i < j \leqq 4$.

(c) $P(A_i \cap A_j \cap A_k) = 1!/4!$, $1 \leqq i < j < k \leqq 4$.

(d) The probability of at least one match is

$P(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - 1/2! + 1/3! - 1/4!$.

(e) Extend this problem so that there n balls in the bucket. Show that the probability of at least one match is

$P(A_1 \cup A_2 \cup \cdots \cup A_n) = 1 - 1/2! + 1/3! - 1/4! + \cdots + (-1)^{n+1}/n!$.

(f) What is the limit of the probability as n increases without bound?