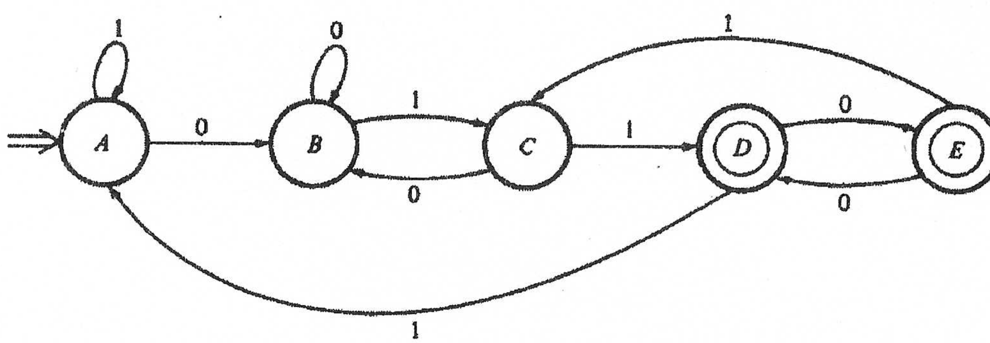
# I. (25%) Answer the following questions.

(a) (5%) Which type of languages does $L = \{a^k b^{k-1} \mid k \geqq 1\}$ belong to?

(b) (10%) Give a grammar that specifies $L = \{a^k b^{k-1} \mid k \geqq 1\}$ with less than 5 production rules.

(c) (10%) Construct a type-3 grammar for the following finite state machine.




# II. (25%) Answer the following questions, 5% for each question.

(a) Given that $a$ and $x$ are integers, $a>1$, $a \mid (11x+3)$, and $a \mid (55x+52)$. Find $a$.

(b) Find the greatest common divisor of 93 and 119 and express it in the form $93m+119n$ for suitable integers $m$ and $n$, where $0<m<119$.

(c) What is the largest possible number of vertices in a graph with 35 edges, all vertices having degree at least 3?

(d) Find a connected graph with as few vertices as possible which has precisely two vertices of odd degree.

(e) Find a connected graph with as few vertices as possible which has precisely two vertices of even degree.

**III. (10%) Fill in the blanks.**

1. The distance from the point $(0, 1, -4)$ to the plane $2(x-1)+6(y-3)+3(z+4)=0$ is ________.

2. Suppose $A = B + C$, where

$$B = \begin{pmatrix} 2 & 6 & 8 \\ -5 & -4 & -2 \end{pmatrix}, \text{ and } C = \begin{pmatrix} -1 & -3 & -7 \\ 7 & 8 & 2 \end{pmatrix}.$$

   Then, the orthogonal complement of $N(A^T)$ is ________.

3. Consider the vector space $C[-1,1]$ with inner product defined by $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$. Then, $||1-x||_2$ is ________.

4. Given the data: $(x, y) = (0,1), (3,4), (6,5)$, the best squares fit by a linear function is ________.

5. The best least squres approximation to $e^x$ on the interval $[0,1]$ by a linear function is ________.

**IV. (15%) Fill in the blanks.**

1. Use Gram-Schmidt process to find an orthonormal basis for $N(A)^{\perp}$, where

$$A = \begin{pmatrix} -1 & 1 \\ 3 & 5 \end{pmatrix}.$$

   Answer = ________.

2. Solve the following initial value problem:

$$\begin{cases} y_1'' &= -2y_1 + y_2 \\ y_2'' &= y_1 - 2y_2 \end{cases} \quad \text{with} \begin{cases} y_1(0) = y_2(0) = 0 \\ y_1'(0) = y_2'(0) = 2 \end{cases}.$$

   Answer = ________.

3. Suppose $A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$. Find $A^6 =$ ________.

4. Suppose $A = \begin{pmatrix} 2 & 1-i \\ 1+i & 1 \end{pmatrix}$. Find a unitary matrix that diagonalizes $A$.

   Answer = ________.

5. Given a function $f(x,y,z) = x^3 + xyz + y^2 - 3x$, determine whether the point $(1,0,0)$ is a local minimum, local maximum, saddle point, or neither of these types.

   Answer = ________.

# V. (10%) Fill in the blanks.

1. Let $H_1, H_2, \cdots, H_n$ be Householder matrices, then $det(H_1 H_2 \cdots H_n) =$ ______.
2. Let $H \in R^{n\times n}$ be a Householder matrix, $\mathbf{x} = [1, 1, 3, 5]^t$, $\|H\mathbf{x}\|_2 =$ ______.
3. Let $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then $\|R_\theta\|_2 =$ ______.
4. Let $R_\theta$ be as defined in (3), the upper bound of $\|R_\theta\|_1 =$ ______.
5. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$, Find $A = LU =$ ______, where $L$ is unit lower-$\Delta$ and $U$ is upper-$\Delta$.

# VI. (15%) Fill in the blanks.

Given

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & -3 \end{bmatrix}.$$

1. The eigenvalues of matrix $A$ are ______.
2. A spectrum decomposition of matrix $A$ is ______.
3. The singular values of matrix $B$ are ______.
4. A singular value decomposition of matrix $B$ is ______.
5. $det(e^B) =$ ______.