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九十三學年度 貴應 系（所） 甲 組碩士班入學考試

科目 線性代數 科號 3103 共 4 頁第 1 頁 *請在試卷【答案卷】內作答

(30%) I. Fill in the blanks.

1. Let

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}.$$

L^{-1}=_________

2. Let $A_n = [a_{ij}] \in R^{n\times n}$ for $n \geq 2$, where

$$a_{ij} = \begin{cases} 1 & if\ i=j\ or\ i=j+1 \\ -1 & if\ i=j-1 \\ 0 & otherwise \end{cases}$$

$det(A_n)$=_________

3. Let $T : R^3 \to R^2$ be a linear transform defined by $T([x,y,z]^t) = ([2x-y, y-2z]^t)$. The kernel of T, $Ker(T)$=_________

4. If B is an $m \times n$ matrix of rank n, then $Null(B)$=_________

5. Let $\mathbf{x} = [1,1,1,1]^t$, $\mathbf{e}_1 = [1,0,0,0]^t$, $\mathbf{u} = \mathbf{x} - \|\mathbf{x}\|_2\mathbf{e}_1$, and define $H = I - \frac{1}{2}\mathbf{u}\mathbf{u}^t$. $det(H)$=_________

(30%) II. Fill in the blanks.

1. Suppose

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   The dimension of the eigenspace corresponding to the eigenvalue $\lambda = 1$ is ________

2. Suppose

$$A = \begin{bmatrix} -2 & -6 \\ 1 & 3 \end{bmatrix}.$$

   Then, $(e^A)^{-1}$ is ________

3. Suppose

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{bmatrix}.$$

   If $B^H B = A$, then B is ________

4. Given the initial value problem: $y''' = 2y'' + 3y' - 2y$, where $y(0) = 3, y'(0) = 2, y''(0) = 6$, the solution $y(t) =$ ________

5. The least squares solution $\hat{\mathbf{x}}$ of the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}, \quad \textit{and} \quad \mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 8 \end{bmatrix},$$

   is ________

(20%) III. Mark ○ if the statement is true, and mark × otherwise.

1. Let **w**, **x**, **y**, **z** be the vectors in R^4. If $\mathbf{w} \perp \mathbf{x}$ and $\mathbf{y} \perp \mathbf{z}$, then U is the orthogonal complement of V, namely, $U = V^{\perp}$, where $U = \text{Span}(\mathbf{w}, \mathbf{y})$, and $V = \text{Span}(\mathbf{x}, \mathbf{z})$.
2. Suppose $A \in R^{m\times n}$ and $B \in R^{n\times r}$. If $C = AB$, $N(B)^{\perp}$ is a subspace of $N(C)^{\perp}$.
3. If **u** and **v** are two vectors in an inner product space V, it is always true that $\|\mathbf{u}+\mathbf{v}\|_2^2 + \|\mathbf{u}-\mathbf{v}\|_2^2 = 2\|\mathbf{u}\|_2^2 + 2\|\mathbf{v}\|_2^2$.
4. Suppose $A \in R^{m\times n}$ and $\hat{\mathbf{x}}$ is a solution to the least squares problem $A\mathbf{x} = \mathbf{b}$. If $\mathbf{y} = \hat{\mathbf{x}} + \mathbf{z}$, for some $\mathbf{z} \in N(A^TA)$, then **y** is another solution.
5. Suppose **x** is a unit vector $\in R^n$. If $H = I - 2\mathbf{x}\mathbf{x}^T$, then H is an orthogonal matrix.
6. Suppose $A \in R^{n\times n}$ and $A^2 = A$. If λ is an eigenvalue of A, then λ must be 1.
7. Suppose $A, B \in R^{n\times n}$. If **x** is the common eigenvector of A and B, then **x** must be an eigenvector of $C = \alpha A + \beta B$, where α and β are two constants.
8. Suppose $A \in R^{n\times n}$. If the columns of A all add up to a fixed constant c, then c is an eigenvalue of A.
9. Suppose A, $B \in R^{n\times n}$. If λ is an eigenvalue of AB, then it is also an eigenvalue of BA.
10. If the eigenvalues of the matrix A are not all distinct, then A is not diagonalizable.

(20%) IV. Choose the best answer in the following questions.

(a) Let $V = \{[a-b, b-c, 0]^t | \ a, b, c \in R\} \subset R^3$, then $dim(V^\perp) =?$

(1) 0, (2) 1, (3) 2, (4) 3, (5) none.

(b) Define $E(a) = I - a\mathbf{e}_3\mathbf{e}_2^t \in R^{n\times n}$, if $a \neq 0$, then the inverse matrix of $E(a)$ is

(1) $E(a^{-1})$, (2) $E(-a^{-1})$, (3) $E(a)$, (4) $E(-a)$, (5) none.

(c) Let $A \in R^{m\times n}$ have rank k and let $CS(A)$ be the column space of A, then $dim(Null(A)) + dim(CS(A)) =?$

(1) m, (2) n, (3) $m-k$, (4) $n-k$, (5) none.

(d) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, the least squares solution of $A\mathbf{x} = \mathbf{b}$ is

(1) $[1,1]^t$, (2) $[-1,-1]^t$, (3) $[0,1]^t$, (4) $[1,0]^t$, (5) none.

(e) Let $Q \in R^{n\times n}$ be orthogonal, then $det(Q) =?$

(1) 1, (2) 1 or -1, (3) -1, (4) n, (5) none.

(f) Let $A \in R^{n\times n}$ have eigenvalues $0, 2, 4, \cdots, 2(n-1)$. Then $trace(A) =?$

(1) n^2, (2) $n(n-1)$, (3) $n(n+1)$, (4) n, (5) none.

(g) Let $V = Span([1,1,1]^t) \subset R^3$, then $dim(V^\perp) =?$

(1) 0, (2) 1, (3) 2, (4) 3, (5) none.

(h) Let $A \in R^{m\times n}$ and $\mathbf{b} \in R^m$, then the condition for $A\mathbf{x} = \mathbf{b}$ must have a solution in R^m is

(1) $m \geq n$, (2) $m < n$, (3) $m = n$, (4) $m \neq n$, (5) none.

(i) Let $L \in R^{n\times n}$ be a unit lower triangular matrix, what is $det(L) + trace(L)$?

(1) 1, (2) n, (3) n+1, (4) n^2, (5) none.

(j) Let $A \in R^{m\times n}$ have $Null(A) = Span(\mathbf{e}_1)$ and $m > n$, what is the rank of A?

(1) n, (2) m, (3) $n-1$, (4) $m-1$, (5) none.