1. (5%) Prove that $n! \geq 2^{n-1}$ for any positive integer n.

2. (10%) Solve the recurrence equation $x_{n+1} - 3x_n + 2x_{n-1} = 3$, $n \geq 1$, with $x_0 = 1$, $x_1 = 2$.

3. (a) (5%) Evaluate $(30)^{16} \bmod 257$

   (b) (5%) Solve for x in $7^x \equiv 1 \pmod{29}$

4. (8 %) Let Fibonacci number $f_0 = 0$, $f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$,

   Prove that $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$

5. Use the following steps to solve the recurrent relation $(n+1)a_{n+1} = a_n + (1/n!)$ for $n \geq 0$

   (a) (3%) Let G(x) be generating function for $\{a_n\}$ show that G(0) = 1 and

   $G'(x) = G(x) + e^x$

   {Note that G'(x) denotes the first derivative of G(x)}

   (b) (3%) Use calculus to show from part (a) that $(e^{-x} G(x))' = 1$

   and conclude that $G(x) = xe^x + e^x$

   {Note that $(e^{-x} G(x))'$ denotes the first derivative of $(e^{-x} G(x))$}

   (c) (4%) Use part(b) to find the closed form for a_n

6. Let G be connected planar simple Graph with e edges and v vertices. Let r be the number of regions in a planar representation of G, and Euler formula $r = e - v + 2$.

   (a) (4%) If G is a connected planar simple graph with e edges and v vertices where $v \geq 3$ and no circuits of length three , prove that $e \leq 2v - 4$

   (b) (3%) Use corollary (a) to show that graph $K_{3,3}$ below is non-planar.
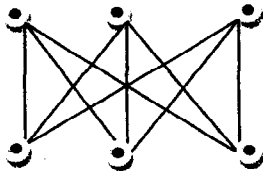
7. (4%) Please fill in the blanks to complete the following recursive function:

```
int Sum(int info[], int fromIndex, int toIndex)
// Compute the sum of the items between fromIndex and toIndex
{
    if (fromIndex _____ toIndex)
        return _____
    else
        return _____
```

Which is the base case? _____

8. (9%) A palindrome is a word, phrase, number or other sequence of units that has the property of reading the same in either direction (the adjustment of punctuation and spaces between words is generally permitted). For example, ignoring the difference between uppercase and lowercase letters, the string "Was it a cat I saw" is a palindrome. Please describe how you could use a *stack* to determine whether a string is a palindrome. You have to give an example.

9. (a) (9%) Using Big Oh notation, indicate the time requirement of each of the following tasks in the worst case. You have to describe all assumptions you make. Note that no assumption will be treated as zero mark.

(1) Each person in a room shakes hands with everyone else in the room.

(2) You climb a flight of stairs.

(3) You read a book thrice.

(b) (3%) Suppose that we plan to implement the Heapsort sorting algorithm in a machine environment where comparison operations are performed at the rate of 2^{25} per second. If we sort a list of 2^{30} items, what is the computation time to be consumed?

10. This question is about operations on binary search tree.

   (a) (5%) Insert 2, 1, 6, 5, 3, 4, 7, 8 in this order into an initially empty binary search tree. Show the results after each insertion.

   (b) (2%) Finally delete 2 and show your result.

11. This question is about bipartite graphs.

   (a) (2%) What is the definition of a bipartite graph.

   (b) (3%) Is it the following graph a bipartite graph? If yes, show the two corresponding partite sets; if no, justify your answer.
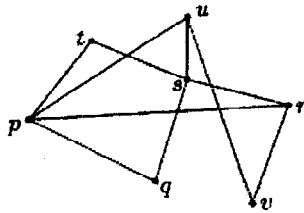
   (c) (3%) Design an algorithm to determine whether a graph G is bipartite.

12. The diameter of a tree is the maximum distance between two vertices, where "distance" means the number of edges on the corresponding path.

   (a) (3%) Find a spanning tree of minimum diameter for the following graph. Please show your spanning tree.

   (b) (7%) Given a connected, undirected graph, design an algorithm for finding a spanning tree of minimum diameter. Prove the correctness of your algorithm.