Let $M(x,y)$ be "x has sent y an e-mail message" and $T(x,y)$ be "x has telephoned y," where the universe of discourse consists of all students in your class. Use quantifiers to express the following statements. (Assume that all e-mail messages that were sent are received.)

(a) (3%) There is a student in your class who has sent everyone else in your class an e-mail message.

(b) (3%) Every student in the class has either received an e-mail message or received a telephone call from another student in the class.

2. (5%) How many nonisomorphic simple graphs are there with 4 vertices? Justify your answer.

3. A rooted tree is called an *m-ary tree* if every internal vertex has no more than m children. A rooted m-ary tree is called a *full m-ary tree* if every internal vertex has exactly m children. A rooted m-ary tree of height h is **balanced** if all leaves are at levels h or h-1. Consider a full m-ary tree T that has 81 leaves and height 4.

(a) (5%) Give the upper and lower bounds for m.

(b) (3%) What is m if T is also balanced?

4. Let f be a function from the set A to the set B. Let S be a subset of B. We define the **inverse image** of S to be the subset of A containing all pre-images of all elements of S. We denote the inverse image of S by $f^{-1}(S)$, so that $f^{-1}(S) = \{a \in A \mid f(a) \in S\}$. Let $g(x) = \lfloor x \rfloor$. Find

(a) (2%) $g^{-1}(\{0\})$.

(b) (2%) $g^{-1}(\{-1, 0, 1\})$.

(c) (2%) $g^{-1}(\{x \mid 0 < x < 1\})$.

5. (7%) Prove that the sum of the cubes of three consecutive positive integers is divisible by 9.

6. (a) (4%) Assume that there are 100 students of different heights, from which two groups of 10 students each are selected. In how many ways can the selection be made so that the tallest student in the first group is shorter than the shortest student in the second group?

(b) (4%) Use a combinatorial argument to show that $C(2n, 2)=2C(n, 2)+n^2$.

7. (6%) Suppose we choose 101 numbers (without repetition) from the integers 1 to 200. Show that, among the chosen numbers, there exist two such that one divides another.

8. (4%) Let a_r be the number of subsets of the set $\{1, 2, \ldots, r\}$ that do not contain two consecutive numbers. Derive a recurrence relation for a_r.

9. (a) (4%) Explain what "a sorting method is stable" means.
   (b) (4%) Is Heap Sort a stable sorting method? If yes, please prove your answer. If no, please give an example to justify your answer.

10. (13%) Show that the number of distinct binary trees with n internal nodes is $\frac{1}{n+1}\binom{2n}{n}$.

11. (4%) A linear list is maintained circularly in an array $q[0,1,\cdots,n-1]$ to represent a circular queue. The variable *front* is used to point to one position from the first element in the circular queue. Another variable *rear* is used to point to the last element in the circular queue. Obtain a formula in terms of *front*, *rear* and n for the number of elements in the circular queue.

12. (4%) Give, using `big oh' notation, the tightest worst case time complexity of the following procedures as a function of n.

```
function recursive (n : integer): integer;
    begin
        if n <= 1   then
            return (2)
        else
            return( 2*recursive(n/2) + 2*recursive(n/2 ))
    end
```

13. (4%) Draw the Max-heap after the following operations are performed beginning with an empty heap: insert 34, insert 39, insert 24, delete-max, insert 5, insert 35, insert 17, delete-max.

14. (4%) Let a hash table and a hashing function be defined as follows:
    var table : array [0..7] of hashnode;
    hashing function: $h(x)$= mid-squaring of x;
    Mids-squaring is defined as:
    1. squaring x
    2. using the second rightmost digit (in octal) of the result in step 1 to obtain the bucket address.
    Let the sequence inserted to the hash table be: insert 23, insert 14, insert 8, insert 18, insert 33. With linear probing as rehash technique, draw the closed hash table.

15. (13%) We have a directed graph $G = (V, E)$ represented using adjacency lists. The edge costs are integers in the range $\{1, 2, 3, 4, 5\}$. Assume that G has no self-loops or multiple edges. Design an algorithm that solves the single-source shortest path problem on G in $O(|V| + |E|)$.