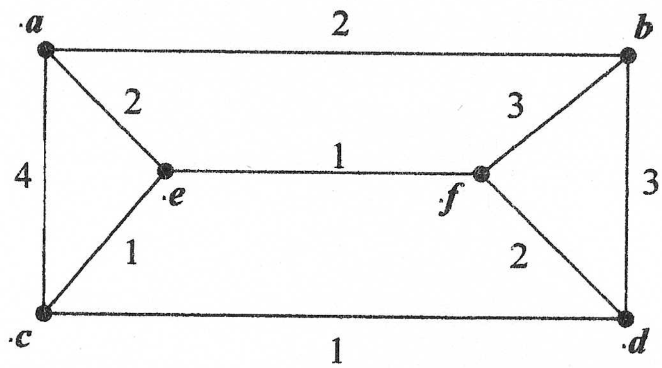
1. Let $S = \{1, 2, 4\}$, and $P(S)$ is the power set of S.
   (a) (2%) Determine the set $T = P(S) - S$.
   (b) (2%) Determine the cardinality $|P(S) \cup P(\{\varnothing\})|$.
2. Express the following statement using predicates and quantifiers. Let $C(x)$ mean that person x works with City Government, and $F(x,y)$ mean that person x knows person y, where x and y are persons in this city.
   (a) (3%) "There is a person in this city who works with City Government but does not knows anyone else in the city."
   (b) (3%) Express the negation of (a) such that there is no implication operator, and no negation is to the left of a quantifier.
3. (5%) Prove that $f(n) = 3^{2n+2} - 5^{n+1}$ is a multiple of 4 for any non-negative integer n.
4. A Mealy machine is a type of finite-state machine that outputs correspond to transitions between states. A Moore machine, on the other hand, is a type of finite-state machine that outputs are determined only by its state.
   (a) (5%) Construct the state diagram of a Mealy machine with minimum number of states to implement an even parity generator where the input x is a one-bit signal, and the output p produces 1 if there are odd number of 1s so far, and 0 otherwise.
   (b) (5%) Construct the state diagram of a Moore machine with minimum number of states to implement the same even parity generator.
5. Consider a randomly generated bit string of length 4. Assume all bits are independent and the probability that a bit is 1 is 0.6.
   (a) (4%) What is the probability that such bit string contains no more than a 0 bit?
   (b) (4%) What is the conditional probability that such string contains at least two consecutive 0s, given that the first bit is a 1?
6. Let the relation R = {(1, 1), (1, 2), (1, 4), (2, 2), (2, 3), (3, 3), (3, 4), (4, 4)} on {1, 2, 3, 4}.
   (a) (4%) Represent this relation by a matrix and a directed graph.
   (b) (3%) Find the matrix that represents the relations R^2. (Note: $R^2 = R \circ R$)

7. Consider the following connected weighted graph.



(a) (5%) Use Dijkstra's algorithm to find the shortest path between the vertices ***a*** and ***d*** in this weighted graph. Give the intermediate results of the step-by-step procedure in Dijkstra's algorithm in your answer.

(b) (5%) Apply Prim's algorithm to find a minimum spanning tree for the above weighted graph. What is the sum of all the edge weights in the computed minimum spanning tree?

8. Suppose we wish to multiply four matrices of real numbers $M_1 \times M_2 \times M_3 \times M_4$ where M_1 is 20 by 10, M_2 is 10 by 40, M_3 is 40 by 100, and M_4 is 100 by 10. Assume that the multiplication of a $p \times q$ matrix by a $q \times r$ matrix requires pqr scalar operations.

(a) (5%) Find the optimal order in which to multiply the matrices so as to minimize the total number of scalar operations.

(b) (13%) Describe an efficient way to find this optimal ordering if there are an arbitrary number of matrices.

9. (5%) Let $T = (V, E)$ be a rooted binary tree and two nodes m and n in V. We can test whether a node m is a proper ancestor of a node n by testing whether m precedes n in X-order but follows n in Y-order, where X and Y are chosen from {pre, post, in}. Determine all those pairs X and Y for which this statement holds.

10. (5%) A linear pattern matching algorithm was proposed by Knuth, Morris, and Pratt. In this algorithm, a failure function (or next table) for the pattern is used to achieve the linear time complexity. Let the pattern be ABAABAB and the position of the pattern start from 1. Give the failure function of the pattern.

11. (5%) Give an n- node Binary Search Tree on which both the *find-min* and *find-max* operations can be executed in constant time. (Clearly identify the smallest and the largest elements in your tree).

12. (4%) Find the order of growth for the following function $f(n)$ in big Oh notation:

```
function f (n : integer) : integer;
var k : integer;
begin
    k = 0;
    while (n > 1 ) do
    begin
        k = 3 * k + 1;
        n = n div 2;
    end
    f = k ;
end
```

13. (13%) Given is a directed graph $G = (V, E)$ represented via adjacency lists and a vertex v_a in V. Design an algorithm that outputs the length of the shortest cycle containing v_a in G. Your algorithm should solve the problem in $O(|V|+|E|)$ time.