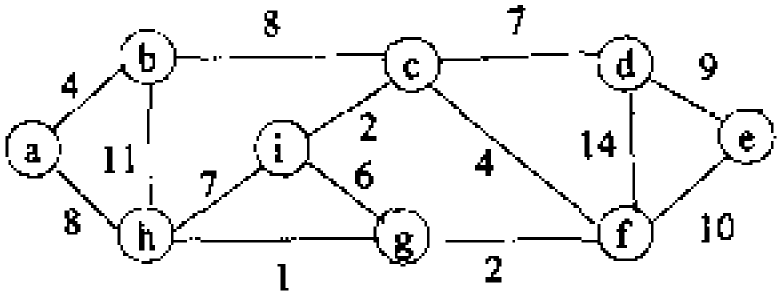
1. (a) (5%) Let $T[1..n]$ be a max-heap of n elements. If an element $T[i]$, $1 \le i \le n$, is assigned a new smaller value, how fast can we adjust the elements in T such that T remains a max-heap? Express your answer in big-O notation and in terms of n and i. Justify your answer.

   (b) (5%) Let A be an array of n elements. Give an $O(n)$ time algorithm to adjust the elements in A such that A becomes a max-heap. Prove that your algorithm does perform in $O(n)$ time.

2. The following is Kruskal's minimum spanning tree algorithm.

```
Kruskal-MST(G=(V,E))
 1.  T=∅
 2.  while T contains less than n-1 edges and E is not empty do begin
 3.     choose an edge (v,w) from E of lowest cost;
 4.     delete (v,w) from E;
 5.     if (v,w) does not create a cycle in T
 6.        then add (v,w) to T                    {select (v,w) as an edge of T}
 7.        else discard (v,w);
 8.  end;
 9.  if T contains fewer than n-1 edges
10.     then writeln('no spanning tree')
11.     else return T;
```

   (a) (5%) If we perform Kruskal-MST on the following graph $G=(V, E)$, which is the 6-th edge added into T?



   (b) (5%) In line 5 of Kruskal-MST, it is needed to determine whether (v,w) creates a cycle in T. Propose an efficient way to implement line 5. Based upon your implementation, how fast can lines 5 and 6 be performed?

3. (5%) Let H be a hash table of size 8 that uses the mid-square function as its hash function. Denote f as the hash function. What's the value of $f(9)$?

4. (5%) Place the following functions into asymptotically ascending order: n, $\log n$, $n/\log n$, $\log n!$, 2^n.

5. (5%) Draw the Max-heap after performing the following operations beginning with an empty heap: insert 13, insert 39, insert 89, insert 56, insert 91, insert 57, delete-max, insert 56, insert 20, delete-max.

6. (5%) Give the best known Θ-notation worst case running time for determining if an undirected graph is a connected component. Explanation is needed.

7. (10%) Give a data structure and use it to write a procedure *TextEditor* to process a line of text. The text editor allows `#' to serve as an *erase* character, which has the effect of canceling the previous uncanceled character. For example, the string *abc#d##e* is really the string *ae*. The editor also allows `&' to serve as a *kill* character, whose effect is to cancel all previous characters.

8. (5%) Let $f(n)=2f(n/3)+4$ for $n=3^k$, $k\geq 1$ and $f(1)=2$. Evaluate $f(729)$.

9. (5%) How many bit strings of length 10 do not contain two consecutive 0s?

10. (5%) A positive integer is perfect if it equals the sum of its divisors other than itself. Prove or disprove that 8128 is perfect.

11. (5%) Let ϕ: $Z^+ \rightarrow N$, and $\phi(n)$ is defined as the number of positive integers less than or equal to n that are relatively prime to n. Find $\phi(2431)$.

12. (a) (2%) Solve $13x \equiv 7 \pmod{31}$. (b) (3%) Evaluate $\sum_{k=1}^{11} k2^k$.

13. (a) (2%) Given that the value of $p\rightarrow q$ is false, determine the value of $(\bar{p}\vee\bar{q})\rightarrow q$.
(b) (3%) Given that the value of $p\rightarrow q$ is true, can you determine the value of $\bar{p}\vee(p\leftrightarrow q)$? Justify your answer.

14. (5%) In how many ways can a group of eight people be divided into committees, subject to the constraint that each person must belong to exactly one committee, and each committee must contain at least two people. (Note that a division into committees of three, three, and two people is considered as the same as a division into committees of two, three, and three people.)

15. (5%) Let R be a binary relation. Let $S=\{(a,b)|(a,c)\in R$ and $(c,b)\in R$ for some $c\}$. Show that if R is an equivalence relation, then S is also an equivalence relation.

16. An ordered n-tuple $(d_1, d_2, \ldots, d_n)$ of nonnegative integers is said to be graphical if there exists a linear graph with no self-loops that has n vertices with the degrees of the vertices being $d_1, d_2, \ldots, d_n$.
(a) (2%) Show that (4, 3, 2, 2, 1) is graphical.
(b) (3%) Show that (3, 3, 3, 1) is not graphical.

17. (5%) Let T_1 and T_2 be two spanning trees of a connected graph G. Let a be an edge that is in T_1 but not in T_2. Prove that there is an edge b in T_2 but not in T_1 such that $(T_1 - \{a\})\cup\{b\}$ and $(T_2 - \{b\})\cup\{a\}$ are spanning trees of G