國 立 清 華 大 學 命 題 紙

八十六學年度 資訊 系（所） ______ 組碩士班研究生入學考試

科目 計算機數學 科號 0802 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

1. (10%) Find the solution to $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

2. (10%) Let $f(n) = 2f(n/2) - 3$ and $f(1) = 5$. Find $f(64)$.

3. (10%) A positive integer is *perfect* if it equals the sum of its divisors other than itself. Prove or disprove that 8128 is perfect.

4. (10%) Let $\circledast$ be a binary operation defined on the set of integers Z by $x \circledast y = x + y - 5$. Prove that $(Z, \circledast)$ is an abelian group.

5. (10%) Show that $(p-1)! \equiv -1 \pmod{p}$ for each prime number p.

6. **(a)** (2%) What is a unitary transformation?
   (b) (3%) List three equivalent conditions which characterize the unitary transformation.
   (c) (5%) Categorize the unitary transformation in R^2 (plane) and describe these transformations geometrically.

7. (10%) Find $\det(A_n)$ if $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 \text{ if } i = j \text{ or } i = j+1, \\ -1 \text{ if } i = j-1, \\ 0 \text{ otherwise.} \end{cases}$$

8. (10%) Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for R^n. Define

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^t + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^t + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^t.$$

Show that A is a symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and that $\mathbf{u}_i$ is an eigenvector corresponding to λ_i for each i.

9. (10%) Find the LU decomposition of the matrix A defined next. (No pivoting is necessary.)

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 7 \end{bmatrix}$$

10. **(a)** (5%) Evaluate the determinant of A^{-1} where

$$A = BCD,$$

and

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 2 & 5 \end{bmatrix}.$$

(b) (5%) Find $A^4 - 5A^3 - A^2$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$