1.(10%) Let $C[a, b]$ be the vector space of all continuous functions on $[a, b]$ with $(f+g)(x) = f(x) + g(x)$, and $(f \cdot g)(x) = f(x) \cdot g(x)$. Let $V$ be the subspace of $C[a, b]$ spanned by $\{1, e^{-x}, e^{x}\}$ and let $D$ be the differentiation operator on $V$.

(a) Find the transition matrix $S$ representing the change of coordinates from $[1, e^{-x}, e^{x}]$ to $[1, \sinh(x), \cosh(x)]$, where $sinh(x) = \frac{e^{x}-e^{-x}}{2}$ and $cosh(x) = \frac{e^{x}+e^{-x}}{2}$.

(b) Find the matrix $A$ representing $D$ with respect to $[1, e^{-x}, e^{x}]$.

2.(15%) Let $A$ be a matrix defined below.

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) Find the eigenvalues and corresponding eigenvectors of matrix $A^5$.

(b) Find the eigenvalues and corresponding eigenvectors of matrix $A^{-1}$.

(c) Evaluate $e^{A}$, where $e^{A}$ is defined as $\sum_{k=0}^{\infty} \frac{A^k}{k!}$.

3.(10%) Let $D_n \in R^{n \times n}$ be defined as

$$D_n(i,j) = \begin{cases} 2 & \text{if } i=j, \\ -1 & \text{if } |i-j|=1, \\ 0 & \textit{elsewhere.} \end{cases}$$

(a) Write down $D_4$?

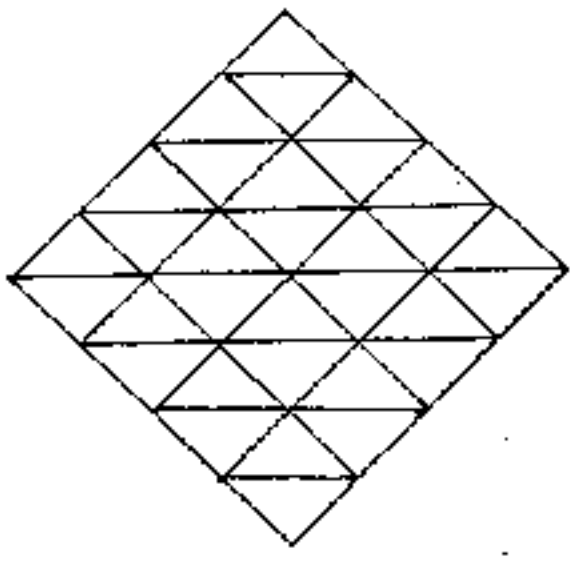
(b) Show that $|D_n| = 2|D_{n-1}| - |D_{n-2}|$ for $n \geq 3$.

(c) Find the determinant of $D_{100}$.

4.(10%) Let $A=(a_{ij}) \in R^{n\times n}$ and $\mathbf{x}=[x_1, x_2, ...., x_n]^t$. Define $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$ and define $\|A\|_1 = \sup_{\|\mathbf{x}\|_1=1} \|A\mathbf{x}\|_1$. Show that $\|A\|_1 = \max_{1\le j\le n} \{\sum_{i=1}^{n} |a_{ij}|\}$.

5.(5%) Let $\mathbf{x}, \mathbf{b} \in R^n$ and $A \in R^{n\times n}$. Define $f(\mathbf{x}) = \mathbf{x}^t A^t A\mathbf{x} - 2\mathbf{b}^t\mathbf{x} + \mathbf{b}^t\mathbf{b}$. Find $\nabla f(\mathbf{x})$, the gradient of $f$.

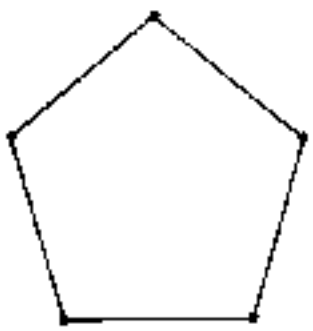
6.(4%) A farmer needs to irrigate the fields in which his crops are growing (A map of the fields is given below in which the fileds are the enclosed areas and edges represent earthen walls between the fields). Because he lacks modern equipment, his method of irrigation is to break holes in the walls and let water from the outside cover the entire field. He wants to irrigate each field and to break as few walls as possible. In how many walls should be break holes? Please give your reason.



7.(6%) Do any two spanning trees for a connected graph always have an edge in common? If so, give a proof. If not, give a counterexample.

8.(9%) The complement of a simple graph $G$ is the simple graph $G'$ with the same vertices as $G$. An edge exists in $G'$ if and only if it does not exist in $G$.

(a) Determine whether the following graph $G$ and its complement are isomorphic. Please give your reason.



(b) Can you find a simple graph $G$ with 6 or 7 vertices such that $G$ and its complement are isomorphic? If so, draw the graph. If not, give your reason.

9.(6%) Let $\phi(x)$ denote the number of positive integers not greater than $x$ and relatively prime to $x$, for examples, $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 2$, $\phi(5) = 4$, $\phi(6) = 2$. Find $\phi(715)$.

八十五學年度 資訊科學 系（所） 組碩士班研究生入學考試

科目 計算機數學 科號 0802 共 4 頁第 4 頁 *請在試卷【答案卷】內作答

10.(10%) Show that among $n+1$ integers less than or equal to $2n$ there are two of them that are relatively prime.

11.(15%) For the graph given below.

(a) Determine the number of directed paths of length 5 that start from vertex $a$ and end at vertex $d$.

(b) Determine the number of directed paths of length 6 that start from vertex $a$ and end at vertex $d$.

(c) Determine the number of directed paths of length $n$ with $n \geq 7$ that start from vertex $a$ and end at vertex $d$.