八十四學年度 資訊科學研究 所 ______ 組碩士班研究生入學考試
科目 計算機數學 科號 0801 共 三 頁第 一 頁 *請在試卷【答案卷】內作答

1. (7%)

(a) Find the matrix that represents reflections through the origin in three dimensions.

(b) Find the matrix that represents reflections through the *xz* plane in three dimensions.

2. (18%)

(a) Find the eigenvalues and a set of eigenvectors of the following matrix, which happens to be degenerate:

$$M = \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(b) Find the spectrum decomposition of matrix M.

(c) Find the eigenvalues and corresponding eigenvectors of M^{-1}.

3. (16%)

Give the probability density function of the random variable X having the following distributions and compute the corresponding *mean* and *variance* for each distribution.

(a) a binomial distribution,

(b) a Poisson distribution,

(c) an exponential distribution,

(d) a normal distribution.

4. (9%)

Let **Y** be a multivariate normal distribution with mean vector **u** and covariance matrix **C** and let **A** be a matrix defined as follows,

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 0.5 & 0 \\ -0.125 & 0.5 \end{bmatrix}.$$

(a) What is the density function of **Y**?

(b) What is the density function of $\mathbf{A}(\mathbf{Y} - \mathbf{u})$?

5. (5%)

What is wrong with the following argument that attempts to show that if R is a relation on a set S that is both symmetric and transitive, then R is also reflexive?

> *Since x R y implies y R x by the symmetric property,*
> *x R y and y R x imply x R x by the transitive property,*
> *thus, x R x is true for each x in S, and so R is reflexive.*

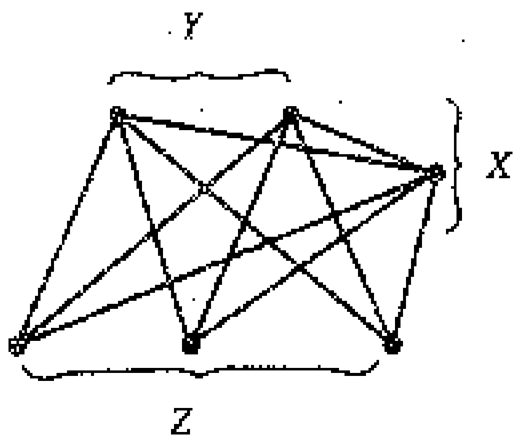
6. (10%)

Is the following argument true or false? Please prove or disprove it.

> Let (A, *) be a group and B a subset of A. If B is a finite set, then (B, *) is a subgroup of (A, *) if * is a closed operation on B.

7. (10%)

The complete tripartite graph $K_{l,m,n}$ is defined as follows: There are three sets of vertices, X, Y, Z, with $|X| = l$, $|Y| = m$, and $|Z| = n$. Two vertices are connected by an edge if and only if they lie in different sets. The graph $K_{1,2,3}$ is illustrated as follows.



(a) How many edges does $K_{l,m,n}$ have?

(b) For what values of l, m, and n is $K_{l,m,n}$ planar?

8. (5%)

Prove the Vandermonde's convolution

$$\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k}=\binom{r+s}{n}.$$

9. (10%)

Use the technique of generating functions to solve the Fibonacci recurrence as follows.
Define the generating function F as

$$F(z)=\sum_{i=0}^{\infty}F_i z^i,$$

where F_i's are the Fibonacci numbers defined by the recurrence:

$$F_0=0,\ F_1=1,\ F_i=F_{i-1}+F_{i-2} \text{ for } i>1.$$

(a) Show that $F(z)=z+zF(z)+z^2F(z)$.

(b) Show that $F(z)=\frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi z}-\frac{1}{1-\hat{\phi}z}\right)$,

where $\phi=\frac{1+\sqrt{5}}{2}$ and $\hat{\phi}=\frac{1-\sqrt{5}}{2}$.

(c) Show that $F(z)=\sum_{i=0}^{\infty}\frac{1}{\sqrt{5}}(\phi^i-\hat{\phi}^i)z^i$.

(d) Prove that $F_i=\phi^i/\sqrt{5}$ for $i>0$, rounded to the nearest integer.

(e) Prove that $F_{i+2}\geq\phi^i$ for $i\geq 0$. That is, Fibonacci numbers grow exponentially.

10. (10%)

Give an asymptotic bound for $T(n)$ in each of the following recurrences, and justify your answers. Assume that T(n) is constant for $n \leq 2$.

(a) $T(n)=T(n-1)+1/n$.

(b) $T(n)=\sqrt{n}T(\sqrt{n})+n$.