國 立 清 華 大 學 命 題 紙

九十二學年度 通訊工程研究 系（所） 甲 組碩士班研究生招生考試

科目 工程數學 科號 3001 共 3 頁第 1 頁 *請在試卷【答案卷】內作答

1. (15%) Let A be a real-valued $n \times n$ matrix. If "A is nonsingular," which of the following statements are true? (Proof is not needed. Simply choose the statements.)

   (a) A^T is invertible.

   (b) The dimension of the nullspace of A is 1.

   (c) The rank of A is n.

   (d) A is diagonalizable.

   (e) The column vectors of A form a basis for $\mathcal{R}^n$, where $\mathcal{R}$ is the set of real numbers.

   (f) The homogeneous system of n linear equations in n unknowns represented by A has nontrivial solutions.

   (g) The determinant of $A^{100} > 0$.

   (h) The dimension of the row space of $A^2 + 9A$ is n.

   (i) The column vectors of A are linearly independent.

   (j) The determinant of $A^2 + 4A = 0$.

   (k) $\lim_{n\to\infty} A^n = \mathbf{0}$.

   (l) The eigenvectors of A span $\mathcal{R}^n$.

   (m) There exists a matrix B that is similar to A.

   (n) A can be a transition matrix with respect to some ordered basis to the standard basis.

   (o) 0 is an eigenvalue of A.

2. (15%) Decide whether the following matrices are positive definite, negative definite, or indefinite?

   (a) $\begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 4 \end{bmatrix}$.

   (b) $\begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$.

   (c) $\begin{bmatrix} 6 & 4 & -2 \\ 4 & 5 & 3 \\ -2 & 3 & 6 \end{bmatrix}$.

3. (20%) Let A be an $n \times n$ matrix with distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let λ be a scalar that is not an eigenvalue of A and let $B = (A - \lambda I)^{-1}$, where I is the identity matrix.

   (a) (10%) Find the eigenvalues of B.

   (b) (10%) Find the eigenspace of B corresponding to each eigenvalue found in (a).

4. (20%) View a random variable as a *vector*. Define the inner product for two vectors (random variables) X and Y as follows:

$$\langle X, Y \rangle = E[XY].$$

   (a) (4%) Verify that $\langle X, Y \rangle$ is indeed an inner product. (Hint: there are three conditions that you need to verify.)

   (b) (4%) Consider a discrete random variable Y with possible values in the set $\{0, 1, 2, \ldots, K\}$ and $P(Y = i) > 0$ for all $i = 0, 1, \ldots, K$. For $i = 0, 1, \ldots, K$, let

$$Y_i = \begin{cases} 1, & Y = i \\ 0, & Y \neq i. \end{cases}$$

   Verify that $\{Y_i, i = 0, 1, 2 \ldots, K\}$ is an *orthogonal* set of vectors.

   (c) (4%) Let $W(Y)$ be the *vector space* spanned by $\{Y_i, i = 0, 1, 2 \ldots, K\}$. Show that $W(Y)$ is the family of random variables that can be represented as a function of Y. (Hint: show that $g(Y)$ is a linear combination of $\{Y_i, i = 0, 1, 2 \ldots, K\}$ for any function g.)

   (d) (4%) Since $\{Y_i, i = 0, 1, 2 \ldots, K\}$ is an *orthogonal* basis of $W(Y)$, the vector projection of a vector X onto $W(Y)$, denoted by $\langle X | W(Y) \rangle$, can be computed by the sum of the vector projection on Y_i, $i = 0, 1, \ldots, K$, i.e.,

$$\langle X | W(Y) \rangle = \sum_{i=0}^{K} \frac{\langle X, Y_i \rangle}{\langle Y_i, Y_i \rangle} Y_i.$$

   Suppose that X is also a discrete random variable with possible values in the set $\{0, 1, 2, \ldots, L\}$ and $P(X = j) > 0$ for all $j = 0, 1, \ldots, L$. Then

$$X = \sum_{j=0}^{L} j X_j$$

where

$$X_j = \begin{cases} 1, & X = j \\ 0, & X \neq j. \end{cases}$$

Show that the vector projection of X on $W(Y)$ is simply the conditional expectation, i.e.,

$$\langle X|W(Y)\rangle = E[X|Y].$$

(e) (4%) It is known that the vector projection $\langle X|W(Y)\rangle$ is the vector in $W(Y)$ that achieves the least square approximation. Use this to show that

$$E[X - E[X|Y]]^2 \leq E[(X - g(Y))^2]$$

for any function g.

5. (15%) Guests arrive at a hotel in accordance with a Poisson process at a rate of six per hour. (Note that the probability mass function of a Poisson random variable X is given by

$$P(X = i) = \frac{e^{-\lambda}\lambda^i}{i!}, \quad i = 0, 1, 2, \ldots$$

where λ is the parameter of the Poisson random variable.)

(a) (5%) What is the probability that no guest arrives during a period of 10 minutes?

(b) (5%) What is the probability that it takes no more than 4 minutes from the arrival of the tenth to the arrival of the eleventh guest?

(c) (5%) Suppose that for the last 10 minutes no guest has arrived. What is the probability that the next guest will arrive in less than 4 minutes?

6. (15%) Let X be a standard normal random variable, i.e., its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2), \quad -\infty < x < \infty.$$

(a) (5%) Find the mean and variance of X^2

(b) (5%) Find the probability density function of X^2.

(c) (5%) Let $\{X_1, X_2, \ldots\}$ be a sequence of independent standard normal random variables. Let $S_n = X_1^2 + X_2^2 + \cdots + X_n^2$. Find

$$\lim_{n\to\infty} P(S_n \leq n + 2\sqrt{2n}).$$