國 立 清 華 大 學 命 題 紙

九十一學年度 通訊工程研究所 甲 組碩士班研究生招生考試

科目 工程數學 科號 3001 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. (15%) The linear operator L defined by $L(p(x)) = p'(x) + p(0)$, where $p'(x)$ denotes the derivative of $p(x)$, maps P_3 into P_2, where P_n is the space of all polynomials of degree less than n.

   (a) (7%) Find the matrix representation of L with respect to the ordered bases $\{x^2,\ x,\ 1\}$ and $\{1,\ 1-x\}$.

   (b) (4%) Let $p(x) = x^2 + 2x - 3$ in P_3; find the coordinates of $L(p(x))$ with respect to the ordered basis $\{1,\ 1-x\}$.

   (c) (4%) Repeat (b) for $q(x) = 4x^2 + 2x$ in P_3.

2. (15%) Given the vector space $C[-1,\ 1]$ with inner product $\langle f,\ g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ and norm $\|f\| = (\langle f,\ f\rangle)^{1/2}$, where $C[-1,\ 1]$ denotes the space of all real-valued functions that are defined and continuous on the closed interval $[-1,\ 1]$.

   (a) (4%) Compute $\langle 1,\ x\rangle$.

   (b) (4%) Compute $\|1\|$ and $\|x\|$.

   (c) (7%) Find the best least squares approximation to $x^{1/3}$ on $[-1,\ 1]$ by a linear function $l(x) = c_1 + c_2 x$.

3. (10%) Solve the following initial value problem:

$$\begin{aligned} y_1' &= 3y_1 + 4y_2 \\ y_2' &= 3y_1 + 2y_2 \\ y_1(0) = 6,&\ \ y_2(0) = 1. \end{aligned}$$

4. (10%) Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

   Find e^A.

國 立 清 華 大 學 命 題 紙

5. (15%) Suppose that by any time t the number of people that have arrived at a train station is a Poisson random variable with mean λt. If the initial train arrives at the station at a time (independent of when the passengers arrive) that is uniformly distributed over $(0, T)$, we are interested in the number of passengers that enter the train. It is assumed that all the passengers that arrive at the station before the initial train would enter the train.

   (a) (7%) Find the mean of this number.

   (b) (8%) Find the variance of this number.

6. (15%) The joint probability density function of random variables X and Y is given by

$$f(x, y) = \frac{1}{8\pi} \exp\left\{-(x^2 + y^2 - 6x + 2y + 10)/8\right\}, \quad -\infty < x < \infty, \quad -\infty < y < \infty.$$

   (a) (5%) Find $P\{X > 3 | Y > 0\}$.

   (b) (5%) Find $E[Y | X \leq 2]$.

   (c) (5%) Find $\mathrm{Var}(X + Y)$.

7. (10%) The moment generating function of a random variable X is given by

$$M_X(t) = E\left[e^{tX}\right] = \left(\frac{2}{2-t}\right)^2.$$

   (a) (5%) Find $E[X]$.

   (b) (5%) Find $\mathrm{Var}(X)$.

8. (10%) If 100 random numbers are selected independently, each uniformly distributed over $(0, 1)$, we are interested in the probability that the sum of these numbers is at least 45. Use Central Limit Theorem to obtain an approximation of this probability. (Express your result in terms of

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

if possible.)