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※請在答案卷內作答

參考用

**Note: Detailed derivations are required to obtain a full score for each problem.**

**1.** (25%) Suppose $A \in \mathbb{R}^{n \times n}$ (the set of $n$ by $n$ real matrices) and $I_n$ is the $n$ by $n$ identity matrix. Answer the following questions using logical reasoning/proof for each case:

**(a)** (5%) The determinant of $A$ is equal to the product of its eigenvalues, i.e., $\det(A) = \prod_{i=1}^{n} \lambda_i$.

**(b)** (5%) If $\lambda$ is an eigenvalue of $A^2$, then $\sqrt{\lambda}$ or $-\sqrt{\lambda}$ is an eigenvalue of $A$.

**(c)** (5%) Assume $A = I_n - 2uu^T$ where $u$ is an $n$ by 1 real unit vector and the superscript '$T$' denotes the transposition of vector or matrix. Show that $A$ is an orthogonal matrix and the eigenvalues of $A$ are $\lambda = \pm 1$.

**(d)** (5%) Assume $A$ is symmetric. For any $t \in \mathbb{R}$ (which is not an eigenvalue of $A$), the eigenvectors of $(A - tI)^{-1}$ are the same as the eigenvectors of $A$, and the corresponding eigenvalues are $(\lambda - t)^{-1}$s, where $\lambda$s are the eigenvalues of $A$.

**(e)** (5%) Suppose $A$ is diagonalizable with only eigenvalues $\lambda = \pm 1$. Prove that $A^2 = I_n$.

**2.** (12%) Let $V$ be a finite-dimensional vector space and $T : V \to V$ be linear. Suppose $\text{rank}(T) = \text{rank}(T^2)$. Prove that $R(T) \cap N(T) = \{0\}$.

**3.** (13%) Given a vector space $V$ over $\mathbb{F}$. Define the dual space of $V^*$ of $V$ as the set of all functions (also known as linear functionals) from $V$ to $\mathbb{F}$, i.e., $V^* \stackrel{\text{def}}{=} \{f | f : V \to \mathbb{F}\}$. It is obvious that $V^*$ is itself also a vector space with the addition $+ : V^* \times V^* \to V^*$ and scalar multiplication $* : \mathbb{F} \times V^* \to V^*$ defined as pointwise addition as well as pointwise scalar multiplication. Given any linear transformation $T : V \to W$. The transpose $T^t$ is a linear transformation from $W^*$ to $V^*$ defined by $T^t(f) = fT$ for any $f \in W^*$. For every subset $S$ of $V$, we define the annihilator $S^0$ as $S^0 \stackrel{\text{def}}{=} \{f \in V^* | f(x) = 0 \; \forall x \in S\}$. Suppose $V, W$ are both finite-dimensional vector spaces and $T : V \to W$ is linear. Prove that $N(T^t) = \big(R(T)\big)^0$.

**4.** (7%) Define a sample space $S$ for the experiment of rolling two fair dice with the maximum number chosen as the desired outcome.

**(a)** (2%) Determine the sample space $S$.

**(b)** (5%) Find the probabilities of all the single-element events.

**5.** (10%) Define a sample space for the experiment of randomly choosing an integer from the interval $[1, 60]$. Let $A$ be a two-interval event that the outcome is within the interval $[1, 15]$ or the interval $[31, 45]$.

**(a)** (4%) If $B$ is a single-interval event with the probability $P(B) = 1/2$ and the maximum element of $B$ is less than 46, determine and explain the dependence/independence of $A$ and $B$.

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**(b)** (3%) If $C$ is the event that the outcome is within the interval $[1, 34]$ and $D$ is the event that the outcome is within the interval $[10, 60]$, determine and explain the dependence/independence of $A$ and $C \cap D$.

**(c)** (3%) Following (b), determine and explain the dependence/independence of $A$ and $C \cup D$.

**6.** (10%) Consider a Bernoulli random variable $X$ with $P(X = 1) = p$ and $P(X = 0) = 1 - p$, and a continuous random variable $Y$ which is conditioned on $X$. The conditional probability distribution function of $Y$ given $X$ is defined as follows: $f_{Y|X}(y|1)$ is a Gaussian distribution with mean $\mu$ and variance $\sigma^2$, and $f_{Y|X}(y|0)$ is an exponential distribution with mean $1/\lambda$.

**(a)** (3%) Find the marginal probability distribution function of $Y$, i.e. $f_Y(y)$.

**(b)** (3%) Find the mean of $Y$.

**(c)** (4%) Find the variance of $Y$.

**7.** (7%) Let the number of defect pixels on an liquid-crystal display (LCD) follow Poisson distribution, and the defect probability of each pixel be $\epsilon$.

**(a)** (2%) For one 1M-pixel LCD monitor, what is the probability that it has no defect pixels? (Note: $1\text{M} = 2^{20}$)

**(b)** (2%) Suppose one company sells 1M-pixel LCD monitors with less than three defect pixels to customers and promises them that they can return the monitor if there is any defect pixel. What is the probability that one such LCD monitor will be returned if every customer carefully examines his/her monitor?

**(c)** (3%) If the company applies the same selling and return policies to 2M-pixel monitors, will the return rate be higher or lower than that of 1M-pixel monitors? Why?

**8.** (10%) Random variable $X$ is uniformly distributed between -1 and 1. Let $X_1, X_2, ...$ be identically-distributed random variables with the same distributions as $X$. Determine which, if any, of the following sequences (all with $i = 1, 2, ...$) are convergent in probability:

**(a)** (4%) $X_i$

**(b)** (3%) $Y_i = \dfrac{X_i}{i}$

**(c)** (3%) $Z_i = (X_i)^i$

**9.** (6%) $X$ is a binomial (5, 0.5) random variable.

**(a)** (3%) Find $P_{X|B}(x)$, where the condition $B = \{X \geq \mu_X\}$

**(b)** (3%) What is $E[X|B]$?