類組：電機類　科目：工程數學 B(3004)　共 2 頁 第 1 頁

※請在答案卷內作答

Note: Detailed derivations are required to obtain a full score for Problem 2 to Problem 10.

1. (15%) Among the 10 statements below, only 5 are true and the other 5 are false. Find out which 5 are true. (You are not obligated to give explanations, but **will get zero point if listing more than 5 of them**).

(a) Let V be a vector space and $S \subseteq \mathsf{V}$ be a subset. Then, $\mathrm{span}(S)$ is the intersection of all subspaces of V that contain S.

(b) Let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be a linear transformation. Let $S = \{v_1, v_2, ..., v_n\}$ be a subset of V. If S is linearly dependent, its image $\mathsf{T}(S)$ is also linearly dependent.

(c) The basis of any vector space uniquely exists.

(d) Let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be a linear transformation. If T is invertible, then $\dim(\mathsf{V}) = \dim(\mathsf{W})$.

(e) Let $A \in \mathsf{M}_{m \times n}(\mathbb{R})$ be an arbitrary matrix. If $m < n$, then $\mathrm{rank}(A) > \mathrm{rank}(A^t)$.

(f) Assume that $A \in \mathsf{M}_{m \times n}(\mathbb{R})$ and $b \in \mathsf{M}_{m \times 1}(\mathbb{R})$. Let x_1 and x_2 be two column vectors in $\mathbb{R}^n$. If $x_1 \neq x_2$ and $Ax_1 = b = Ax_2$, then the system of linear equations $Ax = b$ has infinitely many solutions.

(g) Let A and B be square matrices of the same size. If $AB = O$, then $\mathsf{R}(\mathsf{L}_B) \supseteq \mathsf{N}(\mathsf{L}_A)$. (*Remarks*: L_A and L_B denote the linear transformation of matrix multiplication from the left.)

(h) Let A and B be square matrices of the same size. If $AB = A$, then $B = I$.

(i) Let A be a square matrix and $r \in \mathbb{R}$. Then, $\det(rA) = r\det(A)$.

(j) Assume that $A \in \mathsf{M}_{3 \times 3}(\mathbb{C})$ and $A^t A = -I$. Then, the entries in A cannot all be real numbers.

2. (10%) Define a linear transformation $\mathsf{T}: \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathsf{T}\big((1,0,0)\big) = (0,1,0)$, $\mathsf{T}\big((0,1,0)\big) = (0,0,1)$, and $\mathsf{T}\big((0,0,1)\big) = (1,0,0)$.

(a) (5%) Find a vector $u = (u_x, u_y, u_z)$ such that $\mathsf{T}(u) = u$ and $\sqrt{u_x^2 + u_y^2 + u_z^2} = 1$.

(b) (5%) Is $\mathsf{T}: \mathbb{R}^3 \to \mathbb{R}^3$ one-to-one and onto? Why or why not?

3. (15%) Let $\mathsf{W}_1, \ldots, \mathsf{W}_k$ be subspaces of a vector space V. The **direct sum** V of $\mathsf{W}_1, \ldots, \mathsf{W}_k$ is defined if the following two conditions hold.

$$\mathsf{V} = \sum_{i=1}^{n} \mathsf{W}_i \qquad \text{and} \qquad \mathsf{W}_j \cap \sum_{i \neq j} \mathsf{W}_i = \{0\}\ \forall j\ (1 \leq j \leq k)$$

If the two conditions hold, then V is denoted by $\mathsf{V} = \mathsf{W}_1 \oplus \ldots \oplus \mathsf{W}_k$. Prove or disprove (by providing a counterexample) of the following statements.

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(a) (7%) If $\mathsf{V} = \mathsf{W}_1 \oplus \ldots \oplus \mathsf{W}_k$. Then, for any distinct i and j, W_i and W_j intersect at exactly the zero vector.

(b) (8%) If $\mathsf{V} = \sum_{i=1}^{n} \mathsf{W}_i$, and W_i and W_j intersect at exactly the zero vector for any distinct i, j $(1 \leq i, j \leq k)$. Then V is the direct sum of $\mathsf{W}_1, \ldots, \mathsf{W}_k$.

4. (10%) Let V be a finite-dimensional complex inner product space and $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be a linear operator. T is normal if and only if $\mathsf{TT}^* = \mathsf{T}^*\mathsf{T}$, where T^* is the adjoint of T. Moreover, T is **nilpotent** if there exists $n \in \mathbb{N}$ such that T^n is the zero operator. Prove the following statement. If T is both normal and nilpotent, then T is the zero operator itself.

5. (7%) Prove that if Θ is a random variable from the interval $[0, 2\pi]$, then the dependent variables $X = \sin\Theta$ and $Y = \cos\Theta$ are uncorrelated.

6. (8%) Let X be a random variable; show that for $\alpha > 1$ and $t > 0$, $P(X \geq \frac{\ln\alpha}{t}) \leq \frac{M_X(t)}{\alpha}$, where $M_X(t)$ is the moment generating function of X.

7. (10%) First a point Y is selected at random from the interval $(0, 1)$. Then another point X is selected at random from the interval $(Y, 1)$. Find the probability density function of X.

8. (7%) A coin is tossed twice. Alice claims that the event of two heads is at least as likely if we know that the first toss is a head than if we know that at least one of the tosses is a head.

(a) (4%) Is she correct?

(b) (3%) Does it matter if it is a fair coin or an unfair coin? Compute the exact probabilities for each of the scenarios described above given that we know the coin is a fair coin.

9. (8%) Consider four independent rolls of a 6-sided die. Let X be the number of 1s and let Y be the number of 2s obtained. What is the joint PMF of X and Y?

10. (10%) Alice passes through four traffic lights on her way to work, and each light is equally-likely to be green or red, independent of the others.

(a) (5%) What is the mean and the variance of the number of red lights that Alice encounters?

(b) (5%) Suppose that each red light delays Alice by exactly two minutes. What is the variance of the time by which Alice is delayed by the red lights?

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