類組：電機類　科目：工程數學B(3004)　

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**Note: Detailed derivations are required to obtain a full score for each problem.**

1. (10%) Let $A = \begin{pmatrix} 0 & 3 & 2 & 1 & -4 \\ 2 & 10 & 10 & 16 & 14 \\ -3 & 0 & -5 & -2 & -7 \\ -2 & -1 & -4 & -3 & -6 \\ 2 & 7 & 8 & 11 & 10 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$.

   **(a)** (3%) Compute rank$(A)$.

   **(b)** (2%) Compute rank$(AB)$.

   **(c)** (3%) Compute rank$(A^tAAA^t)$.

   **(d)** (2%) Compute dim$(\mathsf{N}(B^tA))$.

2. (10%) Let V be the vector space spanned by the ordered basis functions $\beta = \{xe^{ax}, e^{ax}, e^{bx}\}$ where $a, b \in R$ and $a \neq b$. Define a linear transformation $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ with parameters $p, q \in R$:

$$\mathsf{T}\,(y(x)) = y'' + py' + qy.$$

   **(a)** (4%) Find the matrix representation for $[\mathsf{T}]_\beta$.

   **(b)** (6%) There are two conditions for $p$ and $q$ such that dim$(\mathsf{N}(\mathsf{T})) = 2$. For each condition, express $p$ and $q$ in terms of $a$ and $b$, and also find the corresponding null space.

3. (5%) Let $A$ and $B$ be $n \times n$ square matrices such that $AB = C$ where $C$ is an upper triangular matrix with $C_{ij} \neq 0$ whenever $j \geq i$. Prove that $A$ and $B$ are both invertible.

4. (16%) Let V be a vector space over a field $\mathbb{F}$, T be a linear operator on V, and W be a subspace of V. We say that W is invariant under T if for each vector $v$ in W the vector T$v$ is also in W. Let W be an invariant subspace for T, and $v \in$ V. The **T-conductor of $v$ into** W, denoted by $\mathsf{S_T}(v, \mathsf{W})$, is defined as the set of all polynomials $g(x)$ over $\mathbb{F}$ such that $g(\mathsf{T})v$ is in W, i.e., $\mathsf{S_T}(v, \mathsf{W}) = \{g(x) \in \mathbb{F}[x] \,|\, g(\mathsf{T})v \in \mathsf{W}\}$.

   **(a)** (8%) Prove the following statement. If W is an invariant subspace for T, then, for each polynomial $g(x) \in \mathbb{F}[x]$, W is invariant under $g(\mathsf{T})$.

   **(b)** (8%) Prove that if W is an invariant subspace for T then $\mathsf{S_T}(v, \mathsf{W})$ is a subspace of $\mathbb{F}[x]$, the set of polynomials over $\mathbb{F}$.

5. (9%) Let T be a linear operator on a finite-dimensional inner product space V. Prove that $\mathsf{N}(\mathsf{T}^*\mathsf{T}) = \mathsf{N}(\mathsf{T})$, where $\mathsf{N}(\mathsf{T})$ is the null space for T.

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**6.** (10%)

(a) (3%) Prove that for any two events $A$ and $B$, we have $P(A \bigcap B) \geq P(A)+P(B)-1$.

(b) (5%) Generalize to the case of $n$ events $A_1, A_2, \cdots, A_n$, by showing that $P(A_1 \bigcap A_2 \bigcap \cdots \bigcap A_n) \geq P(A_1)+P(A_2)+\cdots+P(A_n)-(n-1)$.

(c) (2%) Let $A_1, A_2, \cdots, A_n$ be $n$ events. Show that if $P(A_1)$=$P(A_2)$=$\cdots$=$P(A_n)$=1, then $P(A_1 \bigcap A_2 \bigcap \cdots \bigcap A_n) = 1$.

**7.** (10%) Let $X_0$ be the amount of rain that will fall in the United States on next Christmas day. For $k > 0$, let $X_k$ be the amount of rain that will fall in the United States on Christmas $k$ years later. Let $N$ be the smallest number of years that elapse before we get a Christmas rainfall greater than $X_0$. Suppose that $P(X_i = X_j) = 0$ if $i \neq j$; the events concerning the amount of rain on Christmas days of different years are all independent, and the $X_k$'s are identically distributed. Please show that the probability mass function of $N$ is $P(N=n) = \frac{1}{n(n+1)}$, $n \geq 1$. (Hint: You can calculate $P(N > n)$ first.)

**8.** (5%) Let $\theta$ be a random number between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Find the probability density function of $X = \tan\theta$.

**9.** (5%) The joint PDF of $X, Y$ is the following,

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2}\right\}$$

We know that $X, Y$ are normal random variables. Find $Var(X+Y)$.

**10.** (10%) Let $X_1, X_2, \ldots, X_9$ be independent Poisson random variables with the following PMF:

$$P_{X_i}(k) = \frac{\lambda^k}{k!}e^{-\lambda}, \text{with } \lambda = 1$$

(a) (5%) Use the Markov inequality to obtain a bound on

$$P\left\{\sum_{i=1}^{9} X_i \geq 15\right\}$$

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**(b)** (5%) Use the two-sided Chebyshev inequality to obtain a bound on

$$P\left\{\sum_{i=1}^{9} X_i \geq 15\right\}$$

**11.** (10%) Let $X$ be a random variable with PDF

$$f_X(x) = \begin{cases} x/4, & \text{if } 1 < x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

and let $A$ be the event $\{X \geq 2\}$.

**(a)** (5%) Find $E[X|A]$.

**(b)** (5%) Let $Y = X^2$. Find $E[Y]$ and $var(Y)$.